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Undergraduate Notes on Convolution and Integration by Parts

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***Abstract*---The evaluation of convolution can be cumbersome and unpleasant for most students. In this paper, a novel and much simpler technique is presented. The intent is to convolve two signals given their start and end times. The technique was beneficial as it could easily be explained when it was introduced to undergraduate students in Signals and Systems class. Additionally, a simplistic method on integration by parts by only applying an arrow diagram method (ADM) is explained. The present paper better depicts the methodology with different illustrations.**

***Keywords* ---Convolution, Integration by parts, Signals and Systems**

I. INTRODUCTION

Convolution can be defined as a function generated by integrating two given functions with one being reversed and shifted. Various approaches to evaluating convolution are found in literature [1, 2, and 3], which mitigated some difficulties or mysteries of convolution. However, the literature review involved some considerations that could extensively be broken down for an undergraduate to easily understand the concept of convolution. [1] is a typical textbook that exposes students to the usefulness of convolution in the field of digital signals processing. It underlines the usual graphical approach commonly used. [2] involves step function representations prior to evaluating the convolution integral. However, [3] is much more helpful. It simplifies the evaluation of convolution integrals by giving a shorter and simpler method than that in [2]. Indeed, [3] uses piecewise continuous functions by finding the limits of each fixed-function segment; which in turn characterize the number of integrals to be evaluated. In this paper, a method that convolves two signals given their start and end times is explained. It is similar to the approach used in [3]. However, it directly states the prerequisites and gives the formula of the convolution integral of two given signals.

The second part of the paper deals with a diagram to evaluate an integral using the integration by parts technique. The diagram mitigates the memorization process that leads sometimes to a confusion.

II. DETAILED STEPS FOR CARRYING OUT CONVOLUTION

The convolution integration is defined by,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (1)$$

Where x(t) and h(t) are two given functions or signals, y(t) is the resultant convolution integral, t represents the time variable, and the sign $(*)$ is convolution.

The following are to be defined:

1: t_{h0} = time t at which signal h is turned on

(i.e. h(t) = 0 for all $t < t_{h0}$)

2: t_{h1} = time t at which signal h is turned off

(i.e. h(t) = 0 for all $t > t_{h1}$)

3: t_{x0} = time t at which signal x is turned on

(i.e. x(t) = 0 for all $t < t_{x0}$)

4: t_{x1} = time t at which signal x is turned off

(i.e. x(t) = 0 for all $t > t_{x1}$)

Then the convolution is computed as follows:

$$\begin{cases} 0, & t < t_{h0} + t_{x0} \\ \displaystyle\int_{\max(t_{x0},\, t-t_{h1})}^{\min(t_{x1},\, t-t_{h0})} x(\tau) h(t-\tau) d\tau, & t_{h0} + t_{x0} \le t \le t_{h1} + t_{x1} \\ 0, & t > t_{h1} + t_{x1} \end{cases}$$

It is important to mention that the above interval of integration may need to be further partitioned into subintervals so that the limits of integration for each subinterval can be properly set.

III. EXAMPLES TO ILLUSTRATE CONVOLUTION TECHNIQUE

A. *Example 1:*

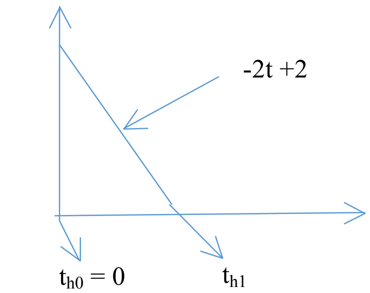


Fig. 1. $h(t) = -2t + 2$

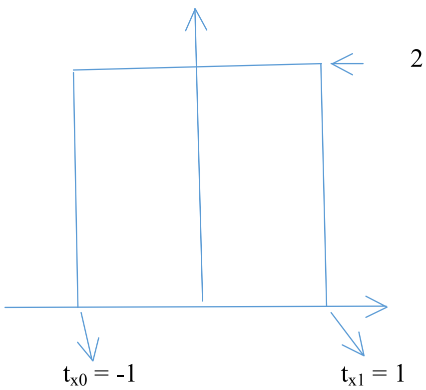


Fig. 2. $x(t) = 2$

The convolution of h(t) and x(t) is given by:

$$y(t) = \begin{cases} 0, & t < 0 - 1 = -1 \\ \int_{\max(-1,t-1)}^{\min(1,t)} 2(-2(t-\tau)+2)d\tau, & -1 \le t \le 1+1 \\ 0, & 2 < t \end{cases}$$

Then

For $-1 \le t < 0$

$$y(t) = \int_{-1}^{t} 4(\tau + (1-t))d\tau = 4\left(\frac{\tau^2}{2} + (1-t)\tau\right)\Big|_{-1}^{t} = -2t^2 + 2$$

For $0 \le t < 1$

$$y(t) = \int_{t-1}^{t} 4(\tau + (1-t))d\tau = 4(\frac{\tau^2}{2} + (1-t)\tau)\Big|_{t-1}^{t} = 2$$

For $1 \le t \le 2$

$$y(t) = \int_{t-1}^{t} 4(\tau + (1-t))d\tau = 4(\frac{\tau^2}{2} + (1-t)\tau)\Big|_{t-1}^{1} = 2(t-2)^2$$

The final answer of the convolution is summarized as,

$$y(t) = \begin{cases} -2t^2 + 2, & -1 \le t < 0 \\ 2, & 0 \le t < 1 \\ 2(t-2)^2, & 1 \le t \le 2 \end{cases}$$

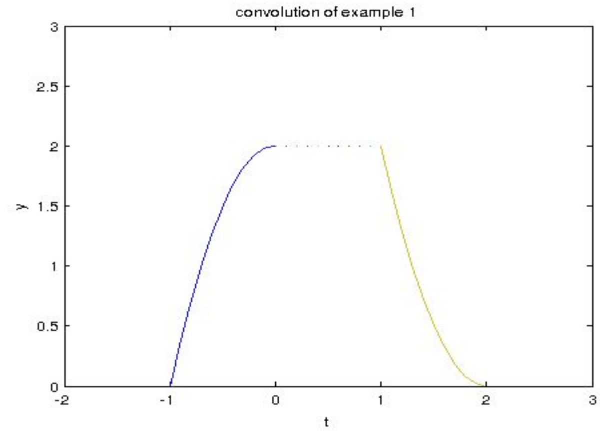


Fig. 3: Convolution plot of example 1

B. *Example 2:*

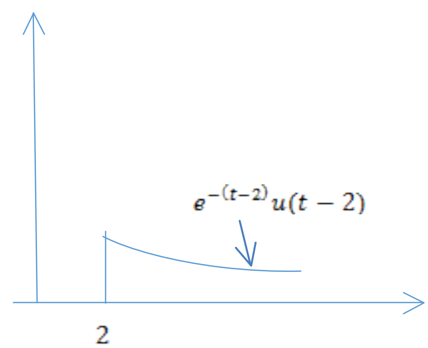


Fig. 4: $h(t) = e^{-(t-2)}u(t-2)$

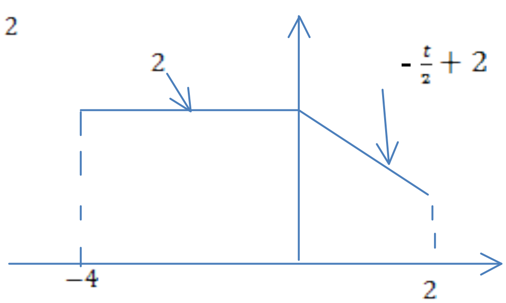


Fig. 5: $x(t) = \begin{cases} 2, & -4 \le t \le 0 \\ -\frac{t}{2} + 2, & 0 \le t \le 2 \\ 0, & elsewhere \end{cases}$

The convolution is evaluated as follows:

$$y(t)=\begin{cases}0, & t<2-4=-2\\ \int_{\max(-4,t-\infty)}^{\min(2,t-2)} x(\tau)h(t-\tau)d\tau, & -2\le t<\infty \\ 0, & elsewhere\end{cases}$$

Similarly as in previous example,

For $-2 \le t < 2$

$$y(t) = \int_{-4}^{t-2} 2e^{\tau-(t-2)}d\tau = 2e^{-(t-2)}e^{\tau}|_{-4}^{t-2} = 2(1-e^{-t-2})$$

For $2 \le t \le 4$

$$y(t) = \int_{-4}^{0} 2e^{\tau-(t-2)}d\tau + \int_{0}^{t-2}\left(\frac{-\tau}{2}+2\right)e^{\tau-(t-2)}d\tau$$

$$= 2e^{-(t-2)}e^{\tau}|_{-4}^{0} + e^{-(t-2)}[\int_0^{t-2} 2e^{\tau}d\tau - \frac{1}{2}\int_0^{t-2} e^{\tau}d\tau]|_0^{t-2}$$

$$y(t) = \frac{7}{2} - \frac{t}{2} + (-2e^{-4} - \frac{1}{2})e^{-(t-2)}$$

For $t \ge 4$

$$y(t) = \int_{-4}^{0} 2e^{\tau-(t-2)}d\tau + \int_{0}^{2}\left(\frac{-\tau}{2}+2\right)e^{\tau-(t-2)}d\tau$$

$$= e^{-(t-2)}[\int_{-4}^{0} 2e^{\tau}d\tau + \int_{0}^{2} 2e^{\tau}d\tau - \frac{1}{2}\int_{0}^{2}\tau e^{\tau}d\tau$$

$$y(t) = 3e^{-t+4} - 2e^{-t-2} + e^{-t+2}$$

Finally, the convolution is:

$$y(t)=\begin{cases}0, & t<-2\\ 2(1-e^{-t-2}), & -2\le t<2\\ \frac{7}{2}-\frac{t}{2}-2e^{-t-2}-\frac{1}{2}e^{-t+2}, & 2\le t<4\\ 3e^{-t+4}-2e^{-t+2}+e^{-t+2}, & t\ge 4\end{cases}$$

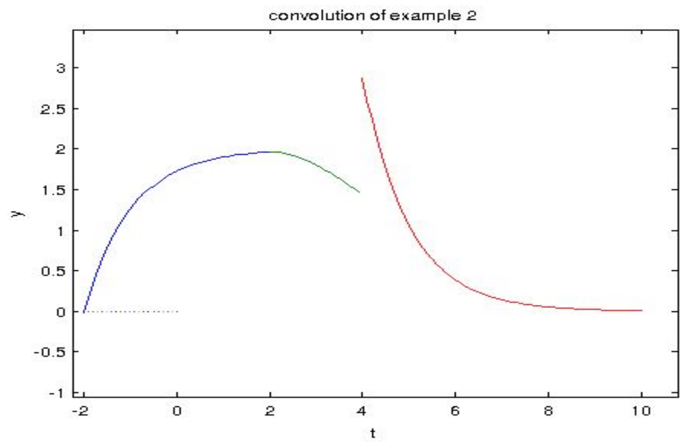


Fig. 6: Convolution plot of example

IV. DETAILED STEPS FOR CARRYING OUT INTEGRATION BY PARTS

To easily evaluate integration by parts requires prior knowledge of the derivative and anti-derivative of functions that form the product of functions. Even after identifying the two prerequisites – derivative and anti-derivative, undergraduate students are confused still when applying the general rule or formula of the integration by parts [4]. The proposed method in this paper is as followed,

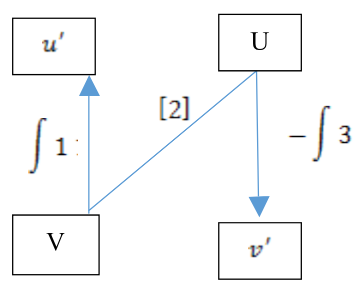


Fig. 7: Arrow Diagram Method (AMD)

Where the arrows indicate the derivatives and the diagonal represents the anti-derivatives. In addition, the derivative and its anti-derivative must be in the same row. The rule is then formulated as,

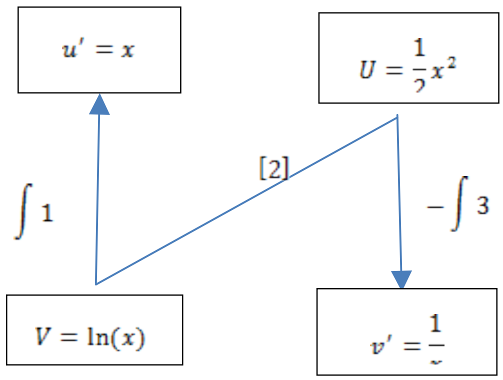
$$\int 1 = [2] - \int 3 \qquad (2)$$

Where $\int 1 = \int u'.V$ and $[2] = V.U$

C. Example 3:

Evaluate $\int x ln(x)\, dx$

Applying the arrow diagram method (AMD) as described in Fig. 7, the result is obtained by:



Applying equation (2),

$$\int x ln(x)\, dx = \frac{1}{2}x^2 ln(x) - \frac{1}{2}\int x dx = \frac{1}{2}x^2 ln(x) - \frac{1}{4}x^2 + constant$$

V. CONCLUSION

The techniques presented in this paper can be used in evaluating the convolution integral and the integration by parts.They can help mitigate some confusions that undergraduate students are facing in the field of signals processing.

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