Analysis of Inelastic Deformations

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Teaching Inelastic Deformation Using Closed-Form Reduced Rigidity Equations

Abstract

When analyzing the deformations of inelastic beams and shafts, current structural analysis procedures that account for the reduced rigidity in the elasto-plastic regions have been limited to simple, idealized conditions. This paper presents a straight-forward approach for teaching this subject to undergraduate students whereby the actual conditions of stress and reduced rigidity in the elasto-plastic regions remain apparent throughout and are applicable over a broad-range of structural conditions. The structural analysis methods that are commonly taught at the undergraduate level to calculate the deflection of beams and the angle of twist of circular shafts are extended for the inelastic condition using closed-form expressions for the reduced rigidity. To assist with teaching the material, a step-by-step procedure is presented for both the elastic and inelastic beam conditions. Several examples illustrate the ease with which the procedure is used, and discussion is provided that highlights the learning opportunities offered by each example.

Introduction

It is now common for Mechanics of Materials textbooks to provide an introduction to inelastic material behavior of structures with axial, flexural and torsional loading conditions. Textbook presentations on the inelastic deformations that result from axial loads are straight-forward and complete over a wide-range of structural conditions, however the methodologies and conditions that can be considered for the flexural and torsional loading cases are limited to simple, idealized situations. Within the field of civil (structural) engineering, design manuals and specifications are moving away from the exclusive use of the allowable stress design method to limit-state design methods that require engineers to understand how structures respond under loading conditions that produce nonlinear material and geometric responses. With structural design philosophies moving in this direction, it is important for educators to develop effective teaching tools that help with this transition that are straight-forward to use and are natural extensions of existing course material.

When dealing with this topic for introductory instructional purposes, textbook authors typically use elastic perfectly-plastic material behavior and rectangular cross-sections for flexural members and solid circular shafts for torsional members. The same approach is taken in this paper because they lend themselves to closed-formed expressions and are therefore ideal for teaching inelastic deformations in a straight-forward manner.

The development of the closed-form reduced rigidity equation for elasto-plastic rectangular beams is discussed first. Examples are presented to illustrate the ease with which this closed-form equation can be used to find inelastic beam deflections. When using this equation with the Virtual Work Method, simple to use closed-form area and centroid formulas of the curvature diagram in the elasto-plastic region are presented for linearly varying moments. Finally, the closed-form reduced rigidity equation for solid circular shafts under elasto-plastic torsional
response is developed, and examples are used to illustrate the ease with which this relationship can be employed to analyze inelastic deformations due to torsion. Simple to use closed-form formulas are presented to obtain the angle of twist of shafts with yielding conditions of constant and linearly varying torque.

**Reduced Rigidity of Rectangular Elasto-Plastic Beams**

A beam will experience a reduction in flexural rigidity when the normal strains due to beam bending are greater than the yield strain, $\varepsilon_y$. For a beam that has elastic, perfectly-plastic material behavior as shown in Figure 1, the stress distribution in Figure 2 will develop for bending moment conditions above the yield moment, $M_y$, but less than the plastic moment, $M_p$. For rectangular cross-sections, with a depth, $h$, and width, $b$, the plastic moment $M_p = 1.5M_y$. The yield moment is the condition that just produces yield stress, $\sigma_y$, at the top and bottom of the beam, and the plastic moment is the condition that produces yield stress over the full depth of the beam. A bending moment between these two values is the elasto-plastic moment, $M_{ep}$.

![Figure 1. Normal stress-strain diagram for elastic, perfectly-plastic material.](image1)

![Figure 2. Normal strain and stress distribution (profile view) due to elasto-plastic moment $M_{ep}$.](image2)
For a given beam with a rectangular cross-section, and specified yield moment \( M_y \) and flexural rigidity \( EI \), the reduced flexural rigidity \( EI_{ep} \) at the location of elasto-plastic moment \( M_{ep} \) is given by the following closed-form expression:\(^{11}\)

\[
EI_{ep} = \left( \frac{M_{ep}}{M_y} \frac{2}{3} \right)^{\frac{3}{2}} EI
\]  

(1)

Appendix A provides a detailed description of the development of Equation (1). The expression in parenthesis varies from 1 to 0 for an elasto-plastic moment \( M_{ep} \) that varies between \( M_y \) and \( M_p \).

**Inelastic Beam Deflection Using the Virtual Work Method**

There are many structural analysis methods taught at the undergraduate level that are used to find the deflection of beams under various types of loads and support conditions. Traditionally these methods have been introduced to students primarily using linear, elastic material properties. Whereas this is a very appropriate thing to do, the methods themselves do not preclude the introduction of flexural rigidities that vary due to yielding over a portion of the beam. Depending upon the load and support conditions of the beam, some methods are more straightforward to use than others. A method that is consistently direct in its approach, and is easily applied over a broad range of beam conditions, is the Virtual Work Method. When introducing the added complexity of reduced beam rigidity that varies over a region of the beam, it is best to use a method that has these characteristics. There are numerous textbooks where the development of the Virtual Work Method is introduced for beams with linear, elastic material behavior, and so it will not be necessary to discuss the entire theory of the method here\(^{12-15}\). The virtual work expression is given as

\[
\delta W = P_e \Delta = \int_0^L \frac{M(x)}{EI} \delta M_v(x) dx
\]

(2)

where \( \delta W \) is the virtual work, \( M(x)/EI \) is the curvature of the beam, and \( \delta M_v(x) \) is the virtual moment that is produced by a virtual load \( P_e \) applied at the location of the desired deflection \( \Delta \). It is assumed in Equation (2) that the actual moments \( M(x) \) and virtual moments \( \delta M_v(x) \) are continuous functions over the length \( L \). When this is not the case over the entire length of the beam, the beam is divided into regions where this remains true and the contributions of virtual work for each region are then summed together. Textbook authors have highlighted a significant feature of the method that expedites the calculations and eliminates the need to evaluate the integral in Equation (2)\(^{13,15}\). Since \( \delta M_v(x) \) is due to a concentrated load and will always be a linear moment, Equation (2) can be written in the following form

\[
\delta W = \int_0^L \frac{M(x)}{EI} (a + bx) dx = a \int_0^L \frac{M(x)}{EI} dx + b \int_0^L \frac{M(x)x}{EI} dx
\]

(3)
Defining $A$ as the area of the curvature diagram over the length $L$, Equation (3) is written as

$$\delta W = aA + bA\bar{x} = A(a + b\bar{x})$$  \hspace{1cm} (4)$$

Recognizing the term in parenthesis is the virtual moment at a specific location of $x$ along the beam (at the centroid of the curvature diagram), the virtual work is obtained very simply by multiplying the area of the curvature diagram by this value of virtual moment.

$$\delta W = A\delta M_v$$  \hspace{1cm} (5)$$

As required by Equation (2), this expression is valid only when both the function for $M(x)$ and the function for $\delta M_v(x)$ are continuous over a particular region of the beam. For typical loading and support conditions, there will likely be several functions for $M(x)$ over the length of the beam. Since the virtual moments are produced from a single virtual force, there will likely be only one or two linear functions for $\delta M_v(x)$. The number of regions and the length over which each region extends can be determined directly and easily from the $M(x)$ and $\delta M_v(x)$ moment diagrams. Referring to Equations (2) and (5), and considering all the regions over the length of the beam, the general expression for the Virtual Work Method is

$$P_iA = \sum_{i=1}^{n} A_i\delta M_{v_i}$$  \hspace{1cm} (6)$$

When initially introducing the use of Equation (6), textbook authors typically limit the loading condition to concentrated and distributed loads such that only linear functions for moment exist. Over each region a constant rigidity $EI$ is used which results in area and centroid formulas for $M(x)/EI$ that are simple triangles and trapezoids. The area $A$ and centroid $\bar{x}$ expressions for these two cases are given in Table 1.

<table>
<thead>
<tr>
<th>Curvature Diagram</th>
<th>Area $A$</th>
<th>Centroid $\bar{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Triangle" /></td>
<td>$M_j L / 2EI$</td>
<td>$2L / 3$</td>
</tr>
<tr>
<td><img src="image" alt="Trapezoid" /></td>
<td>$(M_i + M_j) L / 2EI$</td>
<td>$(M_i + 2M_j) L / 3(M_i + M_j)$</td>
</tr>
</tbody>
</table>

Table 1. Area and centroid formulas for $M/EI$ curvature diagrams with linear moment variation.
Table 2 provides an outline of the Virtual Work Method that is commonly used to determine the deflection at any point along the length of a linear elastic beam. It provides the details on implementing Equation (6) to find the deflection and is given here as a point of reference to later illustrate how the elasto-plastic beam deflection method is incorporated within this procedure.

Table 2. Analysis procedure to determine the deflection of a linear elastic beam.

1. Draw a bending moment diagram for the applied loading condition.
2. Draw a curvature diagram by dividing the moments in step 1 by the beam rigidity $EI$.
3. Apply a virtual force ($P_v = 1$) at the location of the desired deflection and draw the bending moment diagram for this loading condition.
4. Using the curvature diagram and the virtual moment diagram as guides, divide the beam into regions where the two equations are continuous over each region.
5. Using the curvature diagram and the formulas in Table 1, calculate the area and centroid of each region.
6. Using the virtual moment diagram, for each region calculate the virtual moment at the location of the curvature diagram’s centroid.
7. For each region, multiply the area obtained in step 5 by the virtual moment obtained in step 6 and sum the results.
8. The magnitude of the desired deflection is the result obtained in step 7 for $P_v = 1$. (If the magnitude of virtual force is not equal to one, divide the result in step 7 by the magnitude of force used.)
9. If the result obtained in step 8 is a positive number, the direction of the deflection is the same as that of the virtual force. If the result is a negative number, the direction of the deflection is opposite to that of the virtual force.

Depending upon the students’ prior knowledge and experience with the Virtual Work Method, it may be required to go over all the steps in Table 2 and first illustrate the analysis method with linear elastic beams. Students in their first Mechanics of Materials course need to be comfortable
with the procedure in Table 2 before attempting to teach the subsequent material. Students who are already familiar with the Virtual Work Method, and are perhaps in a more advanced course, will find the outline in Table 2 to be a good review of the procedure. An illustrative case is given in Example 1 of a linear elastic beam with two distinct area regions over its length. The steps in Table 2 are easily identifiable with this example, and it serves as a good basis for introducing the elasto-plastic beam analysis procedure later.

**Example 1** The cantilevered beam has the following properties: $E = 29,000 \text{ ksi}$ and $I = 150 \text{ in}^4$. Determine the vertical deflection at the end of the beam using the Virtual Work Method.

It is recognized from the $M/EI$ diagram below that the given loading condition produces two distinct regions with associated areas $A_1$ and $A_2$. Below the curvature diagram is the virtual moment $\delta M_v$ diagram which is developed by placing a virtual force $P_v = 1 \text{ kip}$ at the location of the desired deflection.

The area and centroid of each region are determined using the following formulas for the triangle and trapezoid in Table 1.

$$A_1 = \frac{(-48 \text{ kip ft})(6 \text{ ft})(144 \text{ in}^2/\text{ft}^2)}{2(29,000 \text{ ksi})(150 \text{ in}^4)} = -4.767 \times 10^{-3}$$
\[ \bar{x}_1 = \frac{2}{3} (6 \text{ ft}) = 4 \text{ ft} \]
\[ A_2 = \frac{(48 - 132)(kip \cdot ft)(3 \text{ ft})(144 \text{ in}^2/\text{ft}^2)}{2(29,000 \text{ ksi})(150 \text{ in}^4)} = -8.938 \times 10^{-3} \]
\[ \bar{x}_2 = \frac{(48 + 2 \cdot 132)(kip \cdot ft)}{3(48 + 132)(kip \cdot ft)} (3 \text{ ft}) = 1.73 \text{ ft} \]

For a virtual force \( P_v = 1 \text{ kip} \) acting downward at the end of the beam, the following virtual moments are obtained at the two centroid locations.
\[ \delta \bar{M}_{v_1} = (-1 \text{ kip})(4 \text{ ft}) = -4 \text{ k} \cdot \text{ft} \]
\[ \delta \bar{M}_{v_2} = (-1 \text{ kip})(6 + 1.73)(3 \text{ ft}) = -7.73 \text{ k} \cdot \text{ft} \]

Using Equation (6) with these results, the vertical deflection is found to be
\[ P_v \Delta = A_1 \delta \bar{M}_{v_1} + A_2 \delta \bar{M}_{v_2} \]
\[ (1 \text{ kip})\Delta = (-4.767 \times 10^{-3})(-4 \text{ k} \cdot \text{ft})(12 \text{ in}/\text{ft}) + (-8.938 \times 10^{-3})(-7.73 \text{ k} \cdot \text{ft})(12 \text{ in}/\text{ft}) \]
\[ \Delta = 1.06 \text{ in.} \]

For an elasto-plastic beam that has one or more yielded regions over its length, additional formulas are needed to evaluate the area \( A_i \) and virtual moment \( \bar{M}_{vi} \) in Equation (6). Just as Equation (1) was found to be in closed-form for a rectangular beam with the material properties as described in Figure 1, the area and centroid expressions are also found to be in closed-form.11

For a yield condition that varies over a length \( L_{ep} \) due to a linear distribution of moment between the yield moment \( M_y \) and maximum moment \( M_m \) (for \( M_m < M_p \)), the expression for the area \( A_{ep} \) is
\[ A_{ep} = \frac{M_y^2(1 - \sqrt{3 - 2 \frac{M_m}{M_y}} \frac{L_{ep}}{E I}}{M_m - M_y} \]

The centroid of this area is given as
\[ \bar{x}_{ep} = \frac{(M_y - M_m \sqrt{3 - 2 \frac{M_m}{M_y}})}{3(M_m - M_y)(1 - \sqrt{3 - 2 \frac{M_m}{M_y}})} L_{ep} \]

Appendix B provides a detailed description of the development of Equations (7) and (8).

Since these two equations are functions of just \( L_{ep}, M_y, M_m \) and \( EI \), calculating \( A_{ep} \) and \( \bar{x}_{ep} \) is accomplished in a very straightforward manner for determinate beams with concentrated loads. (For indeterminate beams, moments redistribute after yielding of the cross-section initiates, thus \( L_{ep} \) and \( M_m \) cannot be determined directly and require an iterative approach to obtain the moment diagram.) The area \( A_{ep} \) and centroid \( \bar{x}_{ep} \) expressions from Equations (7) and (8) are given in Table 3.
Table 3. Area and centroid formulas for $M/EI_{ep}$ diagrams with linear moment variation.

<table>
<thead>
<tr>
<th>Curvature Diagram</th>
<th>Area $A_{ep}$</th>
<th>Centroid $\bar{x}_{ep}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{M_y}{EI}$</td>
<td>$\frac{M_y^2(1 - \sqrt{3 - 2 \frac{M_m}{M_y}})}{(M_m - M_y)} \frac{L_{ep}}{EI}$</td>
<td>$\frac{(M_y - M_m \sqrt{3 - 2 \frac{M_m}{M_y}})}{3(M_m - M_y)(1 - \sqrt{3 - 2 \frac{M_m}{M_y}})} \frac{L_{ep}}{EI}$</td>
</tr>
</tbody>
</table>

Table 4 provides an outline of the Virtual Work Method that is proposed by this paper to determine the deflection at any point along the length of a rectangular, elasto-plastic determinate beam with linear varying moments. The parts of the procedure in red font in Table 4 indicate the only additional steps that are necessary to conduct the nonlinear analysis. Students find these additional steps to be very straightforward, and the calculations necessary to obtain the area and centroid in Table 3 are only slightly more complicated than those from Table 1.

Two illustrative cases for determining the deflection of a rectangular, elasto-plastic beam are given in Examples 2 and 3. In Example 2, the beam is loaded in such a way that yielding occurs over a region of the beam adjacent to the fixed end. This example considers two separate regions in a manner that is similar to Example 1, and it highlights for the students the similarities and differences between the two procedures outlined in Tables 2 and 4. Besides having to calculate the length of the yielded region, and having to contend with slightly more complex area and centroid formulas, the methodology employed in Examples 1 and 2 are almost identical.

In Example 3, the beam is loaded in such a way that yielding occurs on both sides of the left support. By considering deflection at the center of the beam, two important features of the method are illustrated. Since the virtual moments only occur over the interior span, the areas and centroids of the curvature diagram for the right and left overhangs do not need to be calculated since they would be multiplied by a zero virtual moment in Equation (6). It is apparent from the bending moment diagram that the moment equation is continuous between the supports, however due to the yielding adjacent to the support and the change in the virtual moment equation at mid-span, the interior span must be divided into three regions. This example involves a good deal of complexity, especially for an inelastic beam bending problem, however it successfully illustrates the method’s versatility and the robustness with which it can accommodate a broad range of concentrated load and support conditions.
Table 4. Analysis procedure to determine the deflection of a rectangular, elasto-plastic beam.
**Example 2** The cantilevered beam has the following properties: \( E = 29,000 \text{ ksi} \), \( b = 3 \text{ in} \), \( h = 8 \text{ in} \) and \( \sigma_y = 36 \text{ ksi} \). Determine the vertical deflection at the end of the beam using the Virtual Work Method.

The moment of inertia for the beam is

\[
I = \frac{3 \text{ in}(8 \text{ in})^3}{12} = 128 \text{ in}^4
\]

Given the beam dimensions and yield stress, the yield moment and plastic moment are found to be

\[
M_y = \frac{I \sigma_y}{h/2} = \frac{(128 \text{ in}^4)(36 \text{ ksi})}{(4 \text{ in})(12 \text{ in/ft})} = 96 \text{ k-ft} \\
M_p = 1.5(96 \text{ k-ft}) = 144 \text{ k-ft}
\]

It is recognized from the moment diagram that a portion of the beam yields, and at the fixed-end the maximum moment \( M_m \) of \(-128 \text{ k-ft} \) is between \( M_y \) and \( M_p \). The length of the yielded region \( L_{ep} \) is determined from the moment diagram using the following relationship.

\[
L_{ep} = \frac{(-128 - (-96)) \text{ k-ft}}{-128 \text{ k-ft}} (10 \text{ ft}) = 2.5 \text{ ft}
\]
The curvature diagram for the given loading condition produces two distinct regions with associated areas $A_1$ and $A_2$. The area of the triangular region $A_1$ and its associated centroid $\bar{x}_1$ are determined in the same manner as given in Example 1.

\[
A_1 = \frac{(7.5 \text{ ft})(-96 \text{ kip} \cdot \text{ft})(144 \text{ in}^2/\text{ft}^2)}{2(29,000 \text{ ksi})(128 \text{ in}^4)} = -1.397 \times 10^{-2}
\]
\[\bar{x}_1 = \frac{2}{3}(7.5 \text{ ft}) = 5 \text{ ft}\]

The area of the yielded region $A_2$ and its associated centroid $\bar{x}_2$ are determined using the formulas in Table 3.

\[
A_2 = \frac{(-96 \text{ kip} \cdot \text{ft})^2(1 - \sqrt{3 - 2(-128)/(-96)})}{(-128 - (-96))k \cdot \text{ft}} \times \frac{(2.5 \text{ ft})(144 \text{ in}^2/\text{ft}^2)}{(29,000 \text{ ksi})(128 \text{ in}^4)} = -1.181 \times 10^{-2}
\]
\[\bar{x}_2 = \frac{(-96 - (-128))\sqrt{3 - 2(-128)/(-96)}(k \cdot \text{ft})}{3(-128 - (-96))\left(1 - \sqrt{3 - 2(-128)/(-96)}\right)(k \cdot \text{ft})} = 2.5 \text{ ft}
\]

For a virtual force $P_v = 1 \text{ kip}$ acting downward at the end of the beam, the following virtual moments are obtained at the two centroid locations.

\[\delta \bar{M}_{v_1} = (-1 \text{ kip})(5 \text{ ft}) = -5 k \cdot \text{ft}\]
\[\delta \bar{M}_{v_2} = (-1 \text{ kip})(7.5 + 1.36)(f t) = -8.86 k \cdot \text{ft}\]

Using Equation (6) with these results, the vertical deflection is found to be

\[
(1 \text{ kip})\Delta = (-1.397 \times 10^{-2})(-5 k \cdot \text{ft})(12 \text{ in}/\text{ft}) + (-1.181 \times 10^{-2})(-8.86 k \cdot \text{ft})(12 \text{ in}/\text{ft})
\]
\[\Delta = 2.09 \text{ in.} \downarrow\]

**Example 3** The beam has the following properties: $E = 200 \text{ GPa}$, $b = 70 \text{ mm}$, $h = 140 \text{ mm}$ and $\sigma_y = 210 \text{ MPa}$. Using the Virtual Work Method, determine the vertical deflection at the mid-point between the two supports.

The moment of inertia for the beam is

\[
l = \frac{0.070 \text{ m}(0.140 \text{ m})^3}{12} = 16.0 \times 10^{-6} \text{ m}^4
\]

Given the beam dimensions and yield stress, the yield moment and plastic moment are

\[
M_y = \frac{l\sigma_y}{h/2} = \frac{(16.0 \times 10^{-6} \text{ m}^4)(210 \times 10^3 \text{ kN/m}^2)}{(0.070 \text{ m})} = 48 \text{ kN} \cdot \text{m}
\]
\[M_p = 1.5(48 \text{ kN} \cdot \text{m}) = 72 \text{ kN} \cdot \text{m}\]
It is recognized from the moment diagram that yielding occurs at the left support, and the maximum moment at this location is below the plastic moment capacity of the section. Referring to the virtual moment diagram and Equation (6), it is noticed that since $\delta M(x) = 0$ over the left and right overhangs, it is not necessary to calculate the areas and centroids for regions 1, 2 and 6.

The length of the yielded region $L_{ep}$ on the right side of the left support is determined from the moment diagram.

$$L_{ep} = \frac{(-60 - (-48)) kN \cdot m}{(-60 - (-24)) kN \cdot m} \cdot 5 m = 1.6 m$$
The area of the yielded region \( A_3 \) and its associated centroid \( \bar{x}_3 \) are determined using the formulas in Table 3.

\[
A_3 = \frac{(-48 \text{ kN} \cdot \text{m})^2 (1 - \sqrt{3 - 2(-60)/(-48)}) (1.6 \text{ m})}{(-60 - (-48))(kN \cdot m)(200 \times 10^6 \text{ kN/m}^2)(16.0 \times 10^{-6} \text{ m}^4)} = -2.929 \times 10^{-2}
\]

\[
\bar{x}_3 = \frac{(-48 - (-60)) \sqrt{3 - 2(-60)/(-48)) (kN \cdot m)(1.6 \text{ m})}{3(-60 - (-48)) \left(1 - \sqrt{3 - 2(-60)/(-48)}\right) kN \cdot m} = 0.8810 \text{ m}
\]

Since there are two separate functions for \( \delta M(x) \) in the interior span, it is necessary to use two regions instead of only one over the elastic portion of the moment diagram. The areas and centroids of regions 4 and 5 are determined in the same manner as given in Example 1.

\[
A_4 = \frac{(-48 + (-42)) (kN \cdot m)(0.83 \text{ m})}{2(200 \times 10^6 \text{ kN/m}^2)(16.0 \times 10^{-6} \text{ m}^4)} = -1.172 \times 10^{-2}
\]

\[
\bar{x}_4 = \frac{(-48 + 2(-42))(kN \cdot m)(0.83 \text{ m})}{3(-48 + (-42))(kN \cdot m)} = 0.407 \text{ m}
\]

\[
A_5 = \frac{(-42 + (-24)) (kN \cdot m)(2.5 \text{ m})}{2(200 \times 10^6 \text{ kN/m}^2)(16.0 \times 10^{-6} \text{ m}^4)} = -2.578 \times 10^{-2}
\]

\[
\bar{x}_5 = \frac{(-24 + 2(-42))(kN \cdot m)(2.5 \text{ m})}{3(-24 + (-42))(kN \cdot m)} = 1.36 \text{ m}
\]

For a virtual force \( P_v = 1 \text{ kN} \) acting upward at the mid-point between the supports, the following virtual moments are obtained at the two centroid locations.

\[
\delta \bar{M}_{v_3} = (0.7857 \text{ m})(-1.25 kN \cdot m)/(2.5 \text{ m}) = -0.3928 kN \cdot m
\]

\[
\delta \bar{M}_{v_4} = (2.093 \text{ m})(-1.25 kN \cdot m)/(2.5 \text{ m}) = -1.046 kN \cdot m
\]

\[
\delta \bar{M}_{v_5} = (1.36 \text{ m})(-1.25 kN \cdot m)/(2.5 \text{ m}) = -0.6818 kN \cdot m
\]

Using Equation (6) with these results, the vertical deflection is found to be

\[
P_v \Delta = A_3 \delta \bar{M}_{v_3} + A_4 \delta \bar{M}_{v_4} + A_5 \delta \bar{M}_{v_5}
\]

\[
(1 \text{ kN}) \Delta = [(-2.929)(-0.3928) + (-1.172)(-1.046) + (-2.578)(-0.6818)](10^{-2})kN \cdot m
\]

\[
\Delta = (0.0413 \text{ m})(1000 \text{ mm/m}) = 41.3 \text{ mm}
\]

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**Reduced Rigidity of Elasto-Plastic Solid Circular Shafts**

A shaft loaded in torsion will experience a reduction in torsional rigidity when the shear strains due to applied torque are greater than the maximum elastic shear strain, \( \gamma_y \). If a shaft has elastic, perfectly-plastic material behavior as shown in Figure 3, the shear stress distribution in Figure 4 will develop for torque conditions above the yield torque, \( T_y \), but less than the plastic torque, \( T_p \).
For a solid circular shaft, $T_p = 4T_y/3$. The yield torque is the torque condition that just produces yield shear stress, $\tau_y$, at outer boundary of the shaft, and the plastic torque is the torque condition that produces yield shear stress over the full cross-section of the shaft. A torque between these two values is the elasto-plastic torque, $T_{ep}$.

![Figure 3](image3.png)

Figure 3. Shear stress-strain diagram for elastic, perfectly-plastic material.

![Figure 4](image4.png)

Figure 4. Shear strain and stress distribution due to elasto-plastic torque $T_{ep}$.

As with the case for inelastic beam bending, in order to investigate the inelastic deformation of circular shafts, the torsional rigidity at any location along the yielded region of the shaft must be known explicitly. The closed-form equation for the reduced torsional rigidity $JG_{ep}$ is given as

$$JG_{ep} = \left( \frac{T_{ep}}{T_y} \sqrt[3]{4 - 3 \frac{T_{ep}}{T_y}} \right) JG$$

(9)

Appendix C provides a detailed description of the development of Equation (9). For a given shaft with a solid circular cross-section, and specified yield torque $T_y$ and torsional rigidity $JG$, the expression in parenthesis varies from 1 to 0 for an elasto-plastic torque $T_{ep}$ that varies between $T_y$ and $T_p$. 
Two illustrative cases for determining the inelastic angle of twist of a solid circular shaft are given in Examples 4 and 5. In Example 4, the yield torque $T_y$ and elastic torsional rigidity $JG$ are specified constants. The shaft is loaded with different concentrated torques such that yielding occurs across the entire length and each region experiences a unique reduction in torsional rigidity. This example illustrates for the students the ease with which the rigidity reduction of Equation (9) can be used in the angle of twist expression

$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{J_{G_{epi}}}$$

(10)

Since the internal torque over each region is constant, the only additional step required before implementing Equation (10) is for the students to use Equation (9) to calculate the elasto-plastic rigidity $J_{G_{ep}}$ of each yielded region based on its corresponding torque $T_{ep}$.

**Example 4** The solid circular shaft has the following properties: $G = 80 \text{ GPa}$, $\tau_y = 149 \text{ MPa}$ and $Diameter = 16 \text{ mm}$. Determine the angle of twist of A relative to D.

The polar moment of inertia and full torsional rigidity of the shaft are

$$J = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.008 \text{ m})^4 = 6.434 \times 10^{-9} \text{ m}^4$$

$$JG = (6.434 \times 10^{-9} \text{ m}^4)(80 \times 10^9 \text{ N/m}^2) = 514.7 \text{ N \cdot m}^2$$

Given the diameter and shear stress at the yield point, the maximum elastic torque and plastic torque are

$$T_y = \frac{\tau_y J}{c} = \frac{(149 \times 10^6)(N/m^2)(6.434 \times 10^{-9} \text{ m}^4)}{0.008 \text{ m}} = 120 \text{ N \cdot m}$$
It is recognized from the torsion diagram that all three regions have torque values between $T_y$ and $T_p$. It is therefore necessary to determine the reduced torsional rigidity of each region using Equation (9).

\[ JG_{AB} = \left( \frac{-140}{-120} \sqrt{4 - 3 \left( \frac{-140}{-120} \right)} \right) (514.7 \text{ N} \cdot \text{m}^2) = 476.6 \text{ N} \cdot \text{m}^2 \]

\[ JG_{BC} = \left( \frac{125}{120} \sqrt{4 - 3 \left( \frac{125}{120} \right)} \right) (514.7 \text{ N} \cdot \text{m}^2) = 512.8 \text{ N} \cdot \text{m}^2 \]

\[ JG_{CD} = \left( \frac{155}{120} \sqrt{4 - 3 \left( \frac{155}{120} \right)} \right) (514.7 \text{ N} \cdot \text{m}^2) = 332.4 \text{ N} \cdot \text{m}^2 \]

Using the appropriate reduced rigidity for each region, the angle of twist is determined in the usual manner.

\[ \phi_{A/D} = \frac{T_{AB}L_{AB}}{JG_{AB}} + \frac{T_{BC}L_{BC}}{JG_{BC}} + \frac{T_{CD}L_{CD}}{JG_{CD}} \]

\[ \phi_{A/D} = \frac{(-140 \text{ N} \cdot \text{m})(0.54 \text{ m})}{476.6 \text{ N} \cdot \text{m}^2} + \frac{(125 \text{ N} \cdot \text{m})(0.39 \text{ m})}{512.8 \text{ N} \cdot \text{m}^2} + \frac{(155 \text{ N} \cdot \text{m})(0.63 \text{ m})}{332.4 \text{ N} \cdot \text{m}^2} \]

\[ \phi_{A/D} = 0.230 \text{ rad} \]

In Example 5, the shaft is loaded in such a way that yielding occurs over a region of a uniformly applied external torque. For this condition of reduced rigidity $JG_{ep}$, it is necessary to evaluate the area of the elasto-plastic region $A_{ep}$ using the following expression.

\[ A_{ep} = \int_0^{L_{ep}} \frac{T_{ep}(x)}{JG_{ep}(x)} \, dx \quad (11) \]

For an internal torque that varies linearly from $T_y$ to $T_m$, and a shaft with a constant torsional rigidity $JG$ over the length $L_{ep}$, the following closed-form expression for the area $A_{ep}$ is given as

\[ A_{ep} = \frac{T_y^2}{2(JG)} \left[ 1 - \left( 4 - 3 \frac{T_m}{T_y} \right)^{2/3} \right] L_{ep} \quad (12) \]

Appendix C provides a detailed description of the development of Equation (12). Students learn from Example 5 two important concepts. First, that even for this more complex yielding condition, a relatively simple closed-form expression is available for this nonlinear analysis.
condition, and second that the concept of the area $A$ used in the Virtual Work Method for beam bending has an application for this type of problem as well. For beam bending, the area of the curvature diagram $M_{ep}/E_{ep}$ is evaluated over the length $L_{ep}$ in Equation (7); it is noticed in Equation (12) that for torsion, the area of $T_{ep}/J_{G_{ep}}$ is evaluated over the length $L_{ep}$ in a very similar manner. Indeed, both equations are dealing with the same concept with regard to Virtual Work. For the case of torsional loading with these two examples, the virtual torque $\delta T_{v}(x) = 1$ and the sum of the areas simply equals to the angle of twist. This concept is also easy to demonstrate to students using Equation (10) where the internal torque is constant over each region. Using Example 4 as an illustration, the virtual torque $\delta T_{v}(x) = 1$ over the entire length and the sum of the rectangular areas equals to the angle of twist.

**Example 5** The solid circular shaft has the following properties: $G = 11,000 \text{ ksi}$, $\tau_y = 4,095 \text{ ksi}$ and Diameter $= 1 \text{ in}$. Determine the angle of twist of A relative to C.
The polar moment of inertia and full torsional rigidity of the shaft are

\[ J = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.5 \text{ in})^4 = 9.817 \times 10^{-2} \text{in}^4 \]

\[ JG = (9.817 \times 10^{-2} \text{in}^4)(11 \times 10^6 \text{ lb}^2/\text{in}) = 1.08 \times 10^6 \text{ lb} \cdot \text{in}^2 \]

Given the diameter and shear stress at the yield point, the maximum elastic torque and plastic torque are

\[ T_y = \frac{\tau_y J}{c} = \frac{(4.095 \times 10^3 \text{lb/in}^2)(9.817 \times 10^{-2} \text{in}^4)}{0.5 \text{ in}} = 804 \text{ lb} \cdot \text{in} \]

\[ T_p = \frac{4}{3} T_y = \frac{4}{3} (804 \text{ lb} \cdot \text{in}) = 1,072 \text{ lb} \cdot \text{in} \]

The length of the yielded region \( L_{ep} \) is determined from the torsion diagram using the following ratio between the maximum torque and the yield torque.

\[ L_{ep} = \frac{(-1,056 - (-804))\text{lb} \cdot \text{in}}{(-1,056 - (-720))\text{lb} \cdot \text{in}} (48 \text{ in}) = 36 \text{ in} \]

Referring to the figure, the angle of twist is determined by summing the three areas \( A_1, A_2 \) and \( A_3 \).

\[ \phi_{A/c} = A_1 + A_2 + A_3 \]

\[ A_1 = \frac{(-720)(24)}{1.08 \times 10^6} = -0.0160 \text{ rad} \]

\[ A_2 = \frac{(-720 - 804)(12)}{2(1.08 \times 10^6)} = -0.0085 \text{ rad} \]

It is recognized from the torsion diagram that region 3 has torque values between \( T_y \) and \( T_p \). It is therefore necessary to determine the area of this region using Equation (10).

\[ A_3 = \frac{(-804)^2 \left[ 1 - \left( 4 - 3 \left( \frac{1,056}{-804} \right) \right)^{2/3} \right]}{2(-1,056 + 804)(1.08 \times 10^6)} (36) \]

\[ A_3 = \frac{(-804)^2 \left[ 1 - \left( 4 - 3 \left( \frac{1,056}{-804} \right) \right)^{2/3} \right]}{2(-1,056 + 804)(1.08 \times 10^6)} (36) \]

\[ A_3 = -0.0362 \text{ rad} \]

\[ \phi_{A/c} = -0.0160 - 0.0085 - 0.0362 \text{ rad} \]

\[ \phi_{A/c} = -0.0607 \text{ rad} \]
References

8. *Building Code Requirements for Structural Concrete* (ACI 318-14), American Concrete Institute, Farmington Hills, MI, 2014.
Appendix A

Recognizing that plane sections remain plane after bending, even after a portion of the beam’s cross-section has yielded, the following relationship for the curvature of the beam is given as

$$\phi = \frac{M_{ep}}{E I_{ep}} = \frac{M_1}{E I} \quad \text{(A.1)}$$

For a given magnitude of moment $M_{ep}$, the beam’s cross-section has a specific value of reduced flexural rigidity, $EI_{ep}$. The bending moment, $M_1$, is the moment that would exist if the beam had not yielded and maintained its full rigidity, $EI$. Although the moment $M_1$ with full rigidity $EI$ does not exist, the strain distribution over the depth of the beam is the same as the actual condition of moment $M_{ep}$ with reduced rigidity $EI_{ep}$. Equation (A.1) is written in the following form in order to solve for $EI_{ep}$ explicitly.

$$EI_{ep} = \left(\frac{M_{ep}}{M_1}\right)EI \quad \text{(A.2)}$$

Of the two moments and two rigidities given above, only $EI$ is known for a given material and cross-section, and only $M_{ep}$ is known for a given moment condition. For beams with a rectangular cross-section, it will be shown that a closed-form equation for $M_1$ can be written in terms of only $M_{ep}$ and $M_y$. Substituting this result for $M_1$ into Equation (A.2), the reduced rigidity $EI_{ep}$ is also found to be a simple closed-form expression that can be used effectively to determine inelastic beam deflections.

Referring to the stress condition in Figure 2, the $M_1$ moment is found considering equilibrium of moments about the centroidal axis. In the figure it is recognized that $M_1$ is comprised of two portions – the $M_{ep}$ moment with stresses at or below $\sigma_y$, and the moment due to the two triangular portions with stresses between $\sigma_y$ and $\sigma_1$. The $M_1$ equation is given as

$$M_1 = M_{ep} + 2 \left[\frac{y_1 b}{2} \left(\sigma_1 - \sigma_y\right) \left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \quad \text{(A.3)}$$

The stresses $\sigma_y$ and $\sigma_1$ are related to one another by the following linear relationship.

$$\frac{\sigma_y}{h/2 - y_1} = \frac{\sigma_1}{h/2} \quad \text{(A.4)}$$

$$\sigma_y = \left(\frac{h - 2y_1}{h}\right) \sigma_1 \quad \text{(A.5)}$$

With $\sigma_y = M_y h/2I$ and $\sigma_1 = M_1 h/2I$, Equation (A.5) can be written in terms of the two moments $M_y$ and $M_1$.

$$M_y = \left(\frac{h - 2y_1}{h}\right) M_1 \quad \text{(A.6)}$$
Solving for $y_1$, the depth of the yielded portion is

$$y_1 = \frac{h}{2} \left( 1 - \frac{M_y}{M_1} \right) \quad (A.7)$$

Substituting these expressions for $\sigma$, $\sigma_1$ and $y_1$ into Equation (A.3), the equation for $M_1$ after simplifying becomes a closed-form expression in terms of only $M_{ep}$ and $M_y$.

$$M_1 = \frac{M_y}{\sqrt{3 - 2 \frac{M_{ep}}{M_y}}} \quad (A.8)$$

This relationship for $M_1$ is substituted into Equation (A.2) to give the closed-form equation for the reduced flexural rigidity $EI_{ep}$:

$$EI_{ep} = \left( \frac{M_{ep}}{M_y} \sqrt{3 - 2 \frac{M_{ep}}{M_y}} \right) EI \quad (A.9)$$
Appendix B

For the condition of reduced rigidity $EI_{ep}$ that varies over a yielded region of the beam, it is necessary to evaluate the area of the elasto-plastic region $A_{ep}$ using the following expression.

$$A_{ep} = \int_{0}^{L_{ep}} \frac{M_{ep}(x)}{EI_{ep}(x)} dx$$  \hspace{1cm} (B.1)

Considering a linear moment variation over the yielded region with length $L_{ep}$, the moments vary between the yield moment $M_{y}$ and the maximum moment $M_{m}$ (for $M_{m} < M_{p}$) according to the following relationship.

$$M_{ep}(x) = M_{y} + \frac{M_{m} - M_{y}}{L_{ep}} x$$  \hspace{1cm} (B.2)

The denominator of Equation (B.1) is written in terms of the elasto-plastic moments that vary over the length of the yielded region.

$$EI_{ep}(x) = \left( \frac{M_{ep}(x)}{M_{y}} \sqrt{3 - 2 \frac{M_{ep}(x)}{M_{y}}} \right) EI$$  \hspace{1cm} (B.3)

Substituting Equations (B.2) and (B.3) into (B.1) yields the following closed-form expression for the area $A_{ep}$ after evaluating the integral and simplifying.

$$A_{ep} = \frac{M_{y}^{2}(1 - \sqrt{3 - 2 M_{m}/M_{y}})}{(M_{m} - M_{y})} \frac{L_{ep}}{EI}$$  \hspace{1cm} (B.4)

The centroid of this area is evaluated using the following expression

$$\bar{x}_{ep} = \frac{1}{A_{ep}} \int_{0}^{L_{ep}} \frac{M_{ep}(x)x}{EI_{ep}(x)} dx$$  \hspace{1cm} (B.5)

Substituting Equations (B.2), (B.3) and (B.4) into (B.5) yields the following closed-form expression for the centroid $\bar{x}_{ep}$ after evaluating the integral and simplifying.

$$\bar{x}_{ep} = \frac{(M_{y} - M_{m} \sqrt{3 - 2 M_{m}/M_{y}})}{3(M_{m} - M_{y})(1 - \sqrt{3 - 2 M_{m}/M_{y}})} L_{ep}$$  \hspace{1cm} (B.6)
Appendix C

Recognizing that plane sections remain plane after twisting, even after a portion of the shaft’s cross-section has yielded, the following relationship for the angle of twist of a circular shaft is given as

\[ \phi = \frac{T_{ep}}{J_{Gep}} = \frac{T_1}{J_G} \]  \hspace{1cm} (C.1)

For a given magnitude of elasto-plastic torque \( T_{ep} \), the shaft’s cross-section has a specific value of reduced torsional rigidity, \( J_{Gep} \). The torque, \( T_1 \), is the torque that would exist if the shaft had not yielded and maintained its full rigidity, \( J_G \). Although the torque \( T_1 \) with full rigidity \( J_G \) does not exist, the shear strain distribution over the cross-section of the shaft is the same as the actual condition of torque \( T_{ep} \) with reduced rigidity \( J_{Gep} \). Equation (C.1) is written in the following form in order to solve for \( J_{Gep} \) explicitly.

\[ J_{Gep} = \left( \frac{T_{ep}}{T_1} \right) J_G \]  \hspace{1cm} (C.2)

Of the two torques and two rigidities given above, only \( J_G \) is known for a given material and cross-section, and only \( T_{ep} \) is known for a given torque condition. It will be shown that a closed-form equation for \( T_1 \) can be written in terms of only \( T_{ep} \) and \( T_y \). Substituting this result for \( T_1 \) in Equation (C.2), the reduced rigidity \( J_{Gep} \) is found to be a simple closed-form expression that can be used effectively to determine the inelastic angle of twist.

Referring to the stress condition in Figure 4, the \( T_1 \) torque is found considering equilibrium of torques about the longitudinal axis. In the figure it is recognized that \( T_1 \) is comprised of two portions – the \( T_{ep} \) torque with shear stresses at or below \( \tau_y \), and the torque due to the triangular portion with shear stresses between \( \tau_y \) and \( \tau_1 \). The equation for \( T_1 \) is given as

\[ T_1 = T_{ep} + \int_{\rho_y}^{c} \left( \frac{\tau_y}{\rho_y} \rho - \tau_y \right) \rho (2\pi \rho) d\rho \]  \hspace{1cm} (C.3)

\[ 2\pi \int_{\rho_y}^{c} \left( \frac{\tau_y}{\rho_y} \rho - \tau_y \right) \rho^2 d\rho = \frac{\pi \tau_y c^4}{2 \rho_y} + \frac{\pi \tau_y \rho_y^3}{6} - \frac{2 \pi \tau_y c^3}{3} \]  \hspace{1cm} (C.4)

The shear stresses \( \tau_y \) and \( \tau_1 \) are related to one another by the following linear relationship.

\[ \frac{\tau_y}{\rho_y} = \frac{\tau_1}{c} \]  \hspace{1cm} (C.5)

With \( \tau_y = 2T_y/\pi c^3 \) and \( \tau_1 = 2T_1/\pi c^3 \), Equation (C.5) can be written in terms of the two torques \( T_y \) and \( T_1 \).

\[ T_y = \left( \frac{\rho_y}{c} \right) T_1 \]  \hspace{1cm} (C.6)

Substituting these expressions for \( \tau_y \), \( \tau_1 \), \( T_y \) and \( T_1 \) into Equation (C.4), the solution to the integral becomes
\[
2\pi \int_{\rho_y}^{c} \left( \frac{t_y}{\rho_y} - t_y \right) \rho^2 d\rho = T_1 + \frac{T_y^4}{3T_1^3} - \frac{4T_y}{3} \quad (C.7)
\]

Substituting this expression into Equation (C.3), the equation for \( T_1 \) after simplifying becomes a closed-form expression in terms of only \( T_{ep} \) and \( T_y \).

\[
T_1 = \frac{T_y}{\sqrt[3]{4 - 3 \frac{T_{ep}}{T_y}}} \quad (C.8)
\]

This relationship for \( T_1 \) is substituted into Equation (C.2) to give the closed-form equation for the reduced torsional rigidity \( JG_{ep} \).

\[
JG_{ep} = \left( \frac{T_{ep}}{T_y} \right)^3 \frac{\sqrt[3]{4 - 3 \frac{T_{ep}}{T_y}}}{JG} \quad (C.9)
\]

For the condition of reduced rigidity \( JG_{ep} \) that varies over a yielded region of the shaft, it is necessary to evaluate the area of the elasto-plastic region \( A_{ep} \) using the following expression.

\[
A_{ep} = \int_0^{L_{ep}} \frac{T_{ep}(x)}{JG_{ep}(x)} \, dx \quad (C.10)
\]

Considering a linear variation of torque over the yielded region with length \( L_{ep} \), the torques vary between the yield torque \( T_y \) and the maximum torque \( T_m \) (for \( T_m < T_p \)) according to the following relationship.

\[
T_{ep}(x) = T_y + \frac{T_m - T_y}{L_{ep}} x \quad (C.11)
\]

The denominator of Equation (C.10) is written in terms of the elasto-plastic torques that vary over the length of the yielded region.

\[
JG_{ep}(x) = \left( \frac{T_{ep}(x)}{T_y} \right)^3 \frac{\sqrt[3]{4 - 3 \frac{T_{ep}(x)}{T_y}}}{JG} \quad (C.12)
\]

Substituting Equations (C.11) and (C.12) into (C.10) yields the following closed-form expression for the area \( A_{ep} \) after evaluating the integral and simplifying.

\[
A_{ep} = \frac{T_y^2}{2(T_m - T_y)} \left[ 1 - \left( 4 - 3 \frac{T_m}{T_y} \right)^{2/3} \right] \frac{L_{ep}}{JG} \quad (C.3)
\]