Solving Problems of Mathematics Accessibility with Process-driven Math: Methods and Implications

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Solving Problems of Mathematics Accessibility with Process-Driven Math: Methods and Implications

Introduction

Engineering is a profession dependent on diversity to serve society. However, the underrepresentation of several groups in engineering continues to present a challenge that affects the profession's ability to serve society. As former president of the National Academy of Engineers, William Wulf, wrote two decades ago, the "profession is diminished and impoverished by a lack of diversity" [1]. Many problems requiring engineering solutions remain unaddressed because the problems themselves are not clearly understood by the engineering community. The people who possess that knowledge are from groups that are underrepresented. Until those groups find representation within the engineering profession, opportunities to advance society with engineering solutions will continue to be lost.

In 2015, a research project with great implications for improving the representation of people with disabilities in engineering started [2]. The goal of the project's current phase is to test the efficacy of Process-Driven Math (PDM), an innovative method of teaching and assessing mathematics. The project originated from the collaboration of three individuals who worked together to solve a specific problem relating to equity in mathematics education. One of these individuals was a university student whose success in college-level mathematics was threatened because the tools required to make math accessible to him did not exist. The student is blind, has significant motor control deficits, and speaks in a whisper. Typical tools used by students who are blind or have low vision were inadequate.

The student entered the university ready to fulfill his academic goals, but a lack of appropriate tools denied him full and equal access to all required course content. On his way to completing a degree in Psychology, mathematics presented a serious obstacle. Unless he could succeed in the Pre-Calculus Algebra course mandated in his degree plan, he would not be able to complete the degree. This student reached out to the academic support services unit at the university and partnered with members of the mathematics tutoring staff to create tools that would allow him to succeed. The outcome of their efforts was the development of PDM. This fully audio method of math instruction and assessment allowed the student, whose motor control deficits precluded his use of braille and math braille, to fully control the solution processes for all of the topics he encountered in his college math courses.

Although PDM originated as a fully audio method for students who are blind or have low vision, it has since been adapted for the benefit of sighted learners, specifically those with print disabilities like dyslexia and dyscalculia. Since the inception of the project, other students with visual and print disabilities at the university have used PDM to make the content in their college mathematics courses accessible. They have successfully completed mathematics courses required for degree completion using the tools provided through PDM.

Cognitive Load and Students with Disabilities

Throughout the development of the method, the goal of keeping the math "process-driven" was the central theme. The name "Process-Driven Math" arose spontaneously from the team's approach of delivering content in a manner that focused on the processes of the mathematics. In the method, numbers, variables, and operation symbols are hidden and revealed in layers and chunks as directed by the student. The process protects a student's working memory from being overloaded, allowing cognitive resources to be more aptly applied to the strategies required to successfully solve the problems.

The foundation of PDM's effectiveness with learners with disabilities is the decrease in cognitive load inherent in to solving complicated mathematics problems. In its development, PDM was intentionally designed around communicating mathematics in manageable pieces to decrease the cognitive load on learners who were overwhelmed by the volume of information inherent in to typical mathematics problems. Cognitive load describes the amount of cognitive resources the active part of human memory can allocate at one time [3]. Often times, the load on cognitive resources is increased by extraneous factors such as distractions like a distant conversation that diverts attention [4]. Conversely, the focus of PDM is on intrinsic factors inherent in to mathematics problems that increase cognitive load, which particularly affect students with visual impairments, learning disabilities like dyscalculia, dyslexia, dysgraphia; and math anxiety [4]. By using PDM, students limit the extra cognitive load created by the design of problems, thereby having more cognitive resources to apply towards the problem-solving process, helping them better master mathematics.

Students with visual impairments face significant challenges in working with mathematics due to the nature of the notational language, which is inherently inaccessible when only available in print or a visual display. Some of the fundamental issues these students face include access to accessible instructional content, the ability to navigate through complex algebraic equations, the ability to perform calculations while manipulating variables, and the ability to complete assignments and take tests in a format that both the student and the instructor can understand [5]. While converting instructional content to mathematics braille is a very effective accommodation for many blind students, this practice alone does not necessarily solve all the issues involved in manipulating expressions and working through multistep problems to finally provide the finished product back to the instructor. Furthermore, not all blind students are proficient in mathematics braille (in the US, either Nemeth Braille Code or Universal English Braille), and it may be unsuitable to some students due to other physical conditions.

In studies comparing the way mathematical equations are commonly read, sighted students typically take a quick glance to comprehend the structure of the equation, and then begin to break down the problem and focus on smaller units of the expression, a process called "chunking" [6]. This technique not only aids in the initial understanding of the expression, but can further be applied to problem solving by determining the value of equation sub-structures such a parenthetical sections and fractions and injecting those derived values back into the equation. This technique is much more difficult for students who are blind to apply due to the linearization of expressions that occur in the conversion to braille, making a quick glance of the equation and subsequent chunking of subexpressions very difficult and increasing cognitive load

[7]. On the other hand, PDM is aimed at reducing this cognitive load by giving the student control over the amount of mathematical information that is revealed at any one time and allowing one to expand or contract individual chunks of information as needed.

The PDM method has been visually adapted for the benefit of students with low vision as well as students with disabilities like dyslexia and dyscalculia. Dyslexia is a language-based learning disability that affects reading, writing, spelling, and organization and is one of the “specific learning disabilities” (as defined by the Individuals with Disabilities Education Act [IDEA]) for which students most commonly seek accommodations [8]. Characteristics associated with dyscalculia include impaired arithmetic fact retrieval, basic numerical processing, working memory, visual-spatial processing and attention [9]. The visually adapted PDM method uses color, shape, and descriptive math vocabulary to chunk the math and reduce cognitive load while simultaneously reinforcing the student’s understanding of the sub-structures of the mathematical expressions. PDM’s visually-displayed expression elements can be covered and then revealed as the student focuses on various aspects of the equation. While students with a visual disability may not be able to benefit from the colors and shapes, they still benefit from the step-by-step process of revealing terms and the use of math vocabulary that establishes communication between instructor and student and teaches the intuition of solving problems. Previous extensions of visual chunking such as this have also shown promise for students with learning disabilities in mathematics classes [10].

Students with learning disabilities or low vision may further benefit from PDM by combining the visual and audio chunking capabilities at the same time. Audio access to mathematics content through computer technology is more recent in usage and has shown promise for students who are visually impaired as well as students with non-visual print disabilities like dyslexia [11]. In addition, future work with PDM will target multi-modal access to equations with enlarged text and refreshable braille displays with audio output.

The goal of PDM is the development of curricula and tools that are customizable to the needs of individuals. The research team is exploring the PDM’s benefit for learners with and without disabilities. For example, math anxiety affects many students, but it is not a diagnosed learning disability. Math anxiety is a psychological condition in which people who are doing mathematics feel tense and agitated in a way that interferes with their ability to do mathematics [12]. It is possible students with math anxiety, and other learners, may also benefit from PDM’s simplification of the presentation of mathematics problems and the provision of discrete, user-controlled steps for the solution processes.

Implications of PDM for Engineering Education

Pedagogy and the Prevalence of Disabilities in the Mathematics Classroom

Students with disabilities are increasingly enrolling in postsecondary education, including college-level engineering programs [13]. According to the National Center for Education Statistics (NCES) [13], 11.1% of college students report having a disability. The Americans with Disabilities Act (ADA) and Section 504 of the Rehabilitation Act require that reasonable accommodations to environments and tasks be made to ensure students with disabilities have an equal opportunity to participate and succeed in postsecondary programs [14]; however, these statutes do not mandate addressing deeper issues of inaccessibility inherent in teaching practices. For example, students with visual impairments might receive preferential seating at the front of the class or a student with a learning disability might get extended time on exams, but the barriers created by the design of the teaching methodology can still remain.

The need to attend to pedagogy is further underscored by the fact that two-thirds of college students with disabilities do not receive accommodations because their institutions do not know they have a disability [15]. Prior to entering college, 60% of students who receive accommodations in high school in compliance with IDEA go on to attend some form of college, a rate similar to students who do not receive accommodations [14]. However, only 34% of students with disabilities complete their degree plan within 8 years, a rate much lower than the 51% of students without disabilities [15].

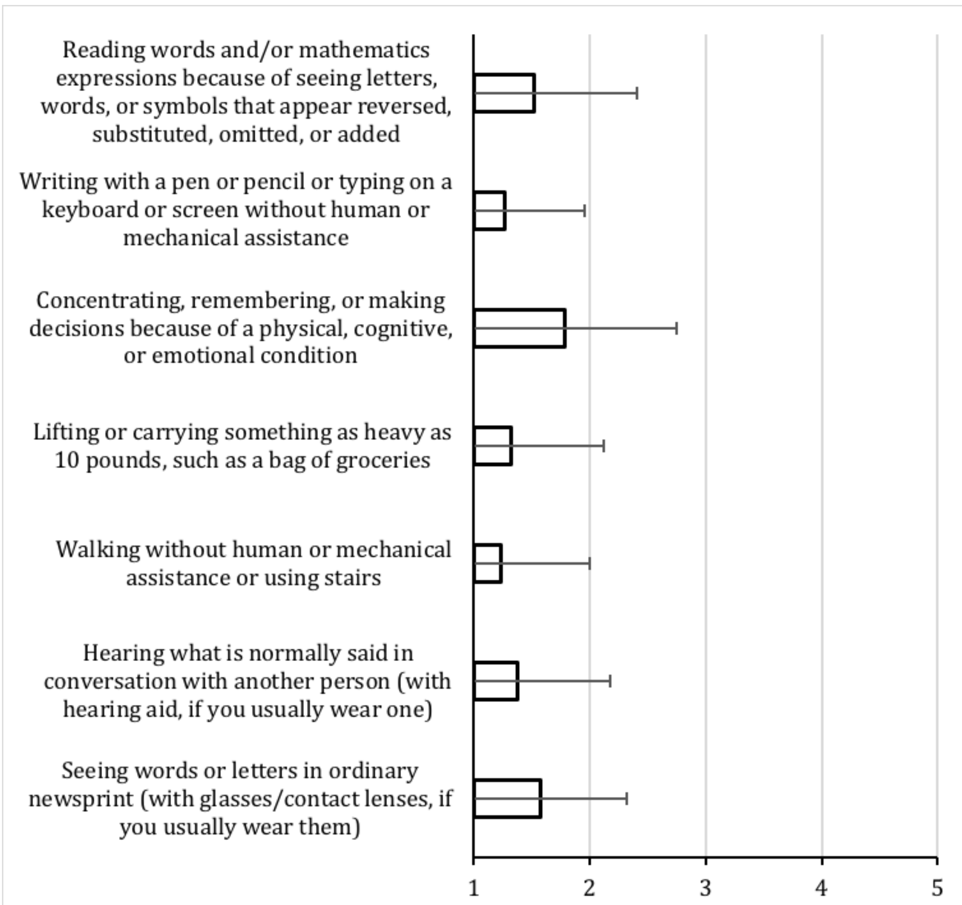


Figure 1. Average survey response levels of difficulties that students experience with whisker for range of one standard deviation.

Recently, the PDM research team started to collect data through a survey of students (N=66) taking College Algebra at four institutions. The survey was used, in part, to gather information about the prevalence of disabilities related to vision, hearing, mobility, and learning. In one set of seven questions, students were asked to identify difficulties that can affect learning, as well as the degree to which those difficulties had affected them. The choice selections included: 1-None, 2-Slight, 3-Moderate, 4-Severe, and 5-Unable to Do. The question was framed using the word "difficulty" instead of "disability" because students without diagnosed disabilities may still notice difficulties that can affect learning.

Figure 1 summarizes the averages and standard deviations of student responses for all seven items. At first glance, the prevalence of difficulties seems rather low, where averages for each item are between none and slight (between a 1 and a 2). However, three of the items concerned with concentration, seeing, and reading stood out because of their larger standard deviations than other items. As shown in Figure 2, these three items had a large minority of responses with levels of difficulty of moderate or higher. Difficulty concentrating had the largest proportion of students with difficulty levels of moderate or higher (27.7%), followed by reading (16.9%), and seeing (15.2%). While difficulty is not the same as disability, it is important to note that five of the seven items had proportions of responses higher than the 11% overall level of students with disabilities in the college classroom and suggest that many students bring difficulties into the classroom that can affect mathematics learning; so the likelihood is strong that engineering instructors will also encounter students who have these difficulties.

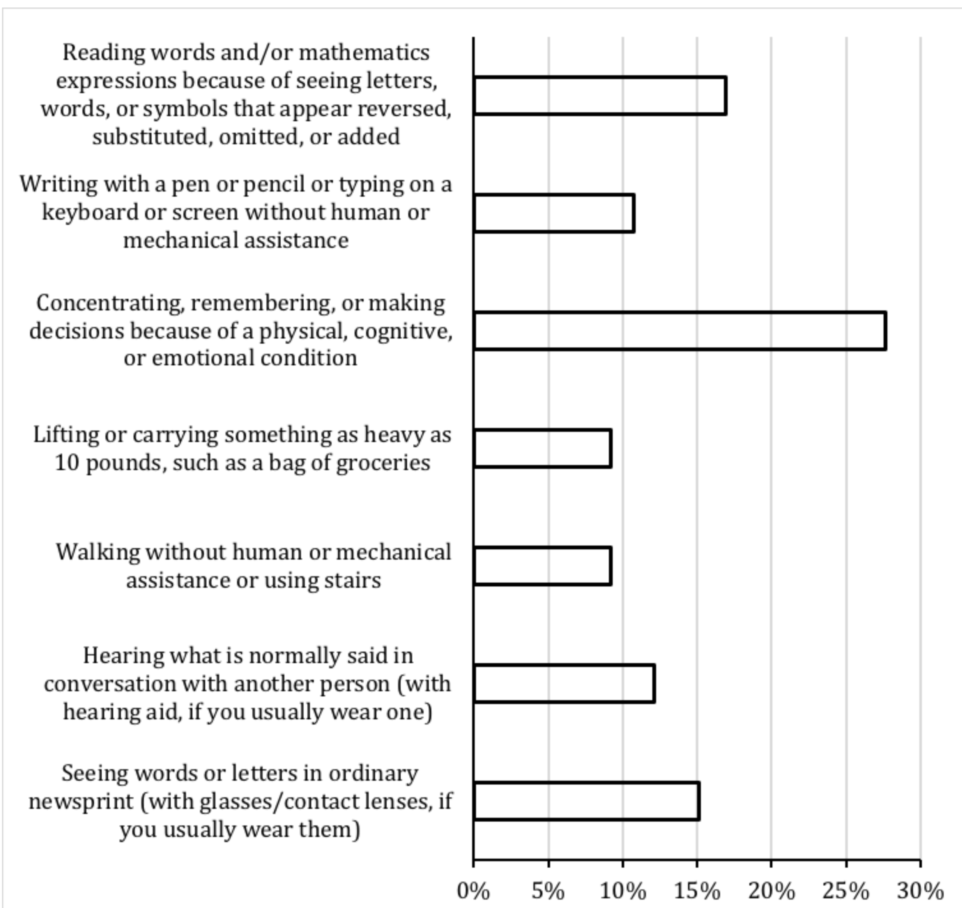


Figure 2. Proportion of responses indicating levels of difficulties of moderate or higher (N=66).

The results of the survey also reveal an important difference between the way students view “disabilities” and the way they view “difficulties.” In the survey, only respondents who declared a difficulty of slight or higher with seeing words or letters in ordinary sizes were asked a qualitatively different question using a Likert-style format with results to the question (N=28) posted in Figure 2 above. We asked students to state their level of agreement (1-Strongly Agree, 2-Agree, 3-Disagree, 4-Strongly Disagree, 5-Do Not Know) with the following statement: I have a disability that affects my ability to learn mathematics. Although a total of N=28 participants indicated they have a difficulty related to seeing, only one participant selected “Strongly Agree” as a response. Of the 27 others who reported some degree of difficulty with vision, 7 disagreed and 16 strongly disagreed that this difficulty affected their ability to learn mathematics. Three respondents indicated that they did not know if their difficulty seeing affected their ability to learn mathematics. The one respondent who strongly agreed with the statement also had moderate to severe difficulties reading and concentrating. It is possible the student was referring to the combination of difficulties reading and concentrating rather than with vision in answering that question. However, there were 13 other cases in which students indicated any difficulty seeing while also indicating moderate to severe difficulties concentrating. Yet in these 13 cases, these students did not select a response that indicated they had a disability that affected mathematics learning. The difference between the number of responses related to students experiencing difficulties, and their perceptions of having a disability that affects their mathematics learning, may point to a greater prevalence of students in post-secondary environments who are not receiving accommodations. In future work, researchers will follow up with case studies to explore the complex relationships between perceptions of disability, accommodations, and academic success.

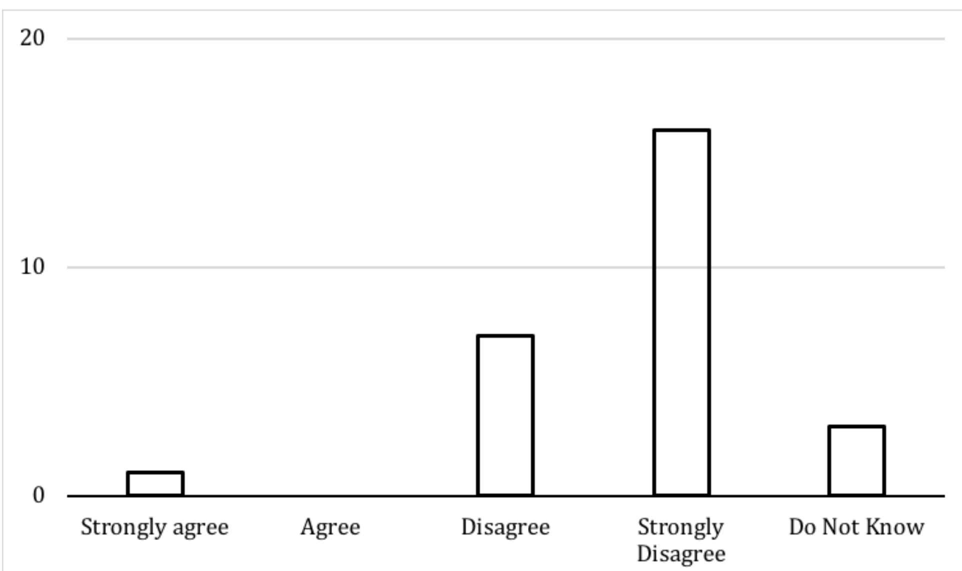


Figure 3. Responses to each level of the Likert-style question: I have a disability that affects my ability to learn mathematics (N=28).

The pattern of underreporting disabilities by students is likely to extend to the engineering classroom as well, making the implications of the design of pedagogy important as well. The principles of PDM have straightforward applications that also serve as tools to make mathematics teaching in engineering more accessible. In the following section, we exemplify a way PDM can be applied with mathematics commonly taught to mechanical engineering students in dynamics courses.

Applications of PDM in the Engineering Classroom for Students with Learning Disabilities and Visual Impairments

A straightforward application of PDM for STEM disciplines such as engineering is to chunk complex problems that create issues for students with low vision, dyslexia, dyscalculia, and math anxiety. For example, mechanical engineering students often encounter problems, such as the proof below taken from a dynamics textbook [16], that contain several variables and operations that are problematic for learners with the disabilities mentioned:

> *2.5 In solving a problem, one person uses cylindrical polar coordinates whereas another uses Cartesian coordinates. To check that their answers are identical, they need to examine the relationship between the Cartesian and cylindrical polar components of a certain vector, say* $\boldsymbol{b} = b_r\boldsymbol{e_r} + b_\theta \boldsymbol{e_\theta}$*. To this end, show that*
>
> $$b_x = \boldsymbol{b}\cdot\boldsymbol{E_x} = b_r\cos\theta - b_\theta\sin\theta, \qquad b_y = \boldsymbol{b}\cdot\boldsymbol{E_y} = b_r\sin\theta + b_\theta\cos\theta.$$

The sample problem contains many complex features that may increase barriers for some students. Examples include:

- variables with subscripts that are issues for students who are blind or visually impaired because of the inherently small sizes of subscripts (e.g., b_x);
- symbols, numbers, and operations that are easy to confuse for people with dyslexia/dyscalculia (e.g., θ and 0 or + and ÷); and
- lengthy equations with numerous operators, functions, and coordinate systems that may increase apprehension for students with math anxiety (e.g., arithmetic and vector operators, Cartesian and cylindrical polar coordinate systems, and sine and cosine trigonometric functions).

The graphic representations in the section that follows demonstrate the manner in which chunking breaks the above problem down into more manageable pieces that reflect the logic of the mathematical substructures. Initially, the chunks are labeled with appropriate math vocabulary, temporarily hiding the numbers, variables, and symbols to deliver only the broad landscape of the problem. This first step functions as an instructional pause button that gives students additional time to formulate a strategy before working memory is taxed with the details of the problem. The full content of the problem is then revealed successively in layers. As the first layer of vocabulary is peeled away, smaller chunks are revealed with a vibrant but simplified visual display. Color and shape are used to associate analogous mathematical structures while simultaneously creating visual distinction between adjacent elements in the problem.

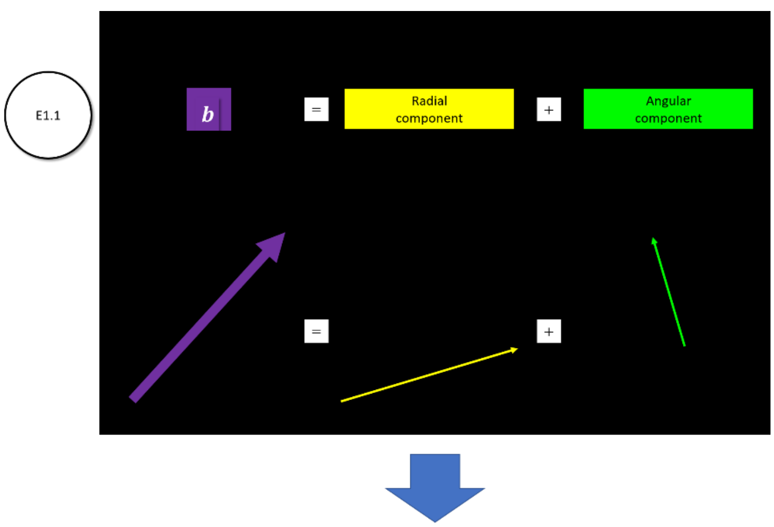
In PDM, encoding is the process of building the structure of the problem from the ground up using vocabulary, shape, and color to chunk the problem. PDM decoding refers to a series of transformations using vocabulary, shape, and color to simplify or solve the problem one chunk at a time. Within the body of the paper, the encoding as well as the first phase of decoding for the problem above are shown. (See the Appendix for the entire encoding and decoding processes for this problem.)

First Phase of the Encoding Process:

E1.1. The equation for vector ***b*** is comprised of three rectangles representing the three terms of the equation. Vector ***b*** is represented in both its equation form and its geometric form. In its equation from, vector ***b*** is a purple square labeled with a bold, italicized letter "b." It is set equal to the radial and angular components of the vector that are colored rectangles. The yellow rectangle is labeled "radial component" and the green rectangle is labeled "angular component". The geometric form of the equation is lined up directly below the original equation and uses arrows to represent each term in the vector. Arrows are the same color as their analogous terms directly above them.

E1.2. The radial and angular components are each shown to contain a circle and a triangle. Consistency with color is maintained and yellow is again used to signify the radial component and green is again used to signify the angular component. The circles represent the scalar magnitude of each component and the triangles represent the unit direction of each component.

E1.3. The component terms with subscripts of vector ***b*** are then revealed to represent the full complexity of the vector in polar coordinate form.



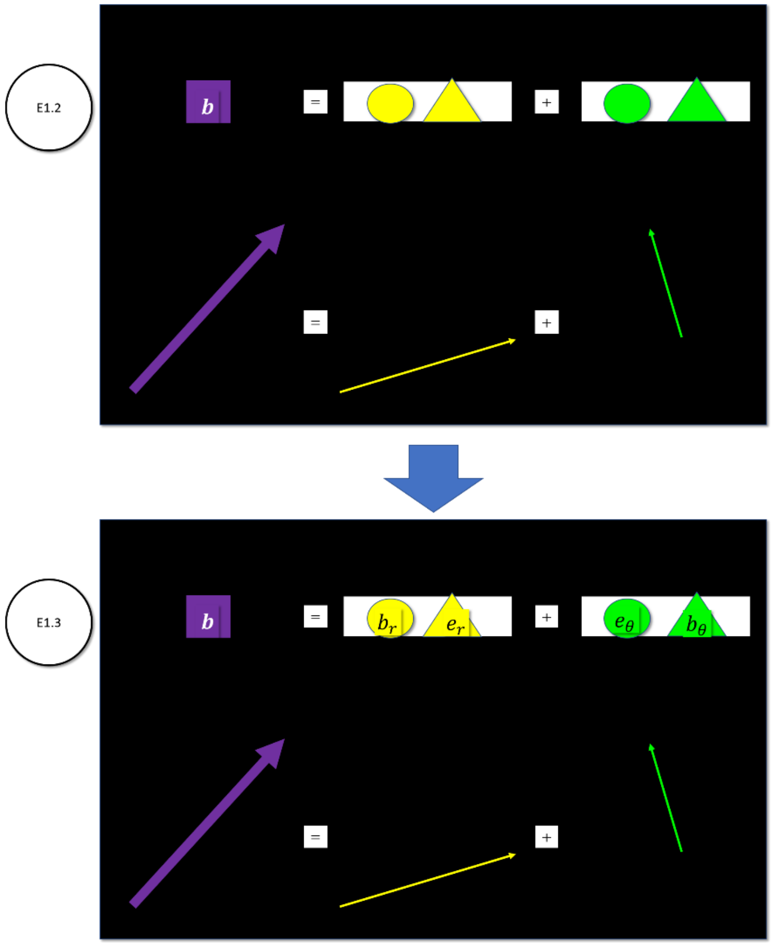


Figure 4. The first phase of the PDM encoding process applied to the dynamics problem.

Second Phase of the Encoding Process:

E2.1. - E2.3. The steps closely resemble E1.1 - E1.3 using a different color scheme. Red and blue colors are used to represent the component vectors in the Cartesian reference frame. Only step E2.3 is shown in Figure 5 to conserve space.

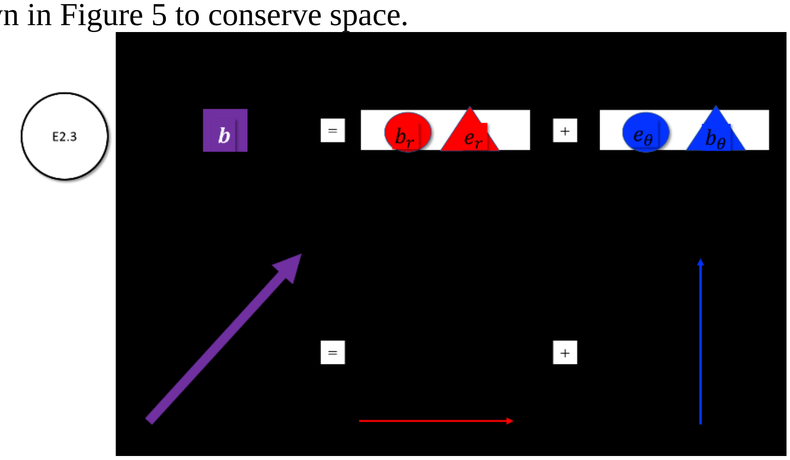


Figure 5. The second phase of the PDM encoding process applied to the dynamics problem.

First Phase of the Decoding Process:

To solve the problem, the cylindrical polar coordinate components must be transformed into a system of Cartesian coordinate components using trigonometric functions. The same shapes and colors used to represent previous terms above are used to maintain continuity in a six-step process. The equation forms of each step are also paired with the geometric representation, again to make clear the spatial relationship inherent in the vector transformation:

D1.1. The transformation for the radial direction unit vectors of ***b*** into x- and y-unit direction vectors is represented by an equation with three terms. The trigonometric functions are represented by two vectors intersecting at their tails.

D1.2. The name for each component is then revealed as well as the specific trigonometric function needed.

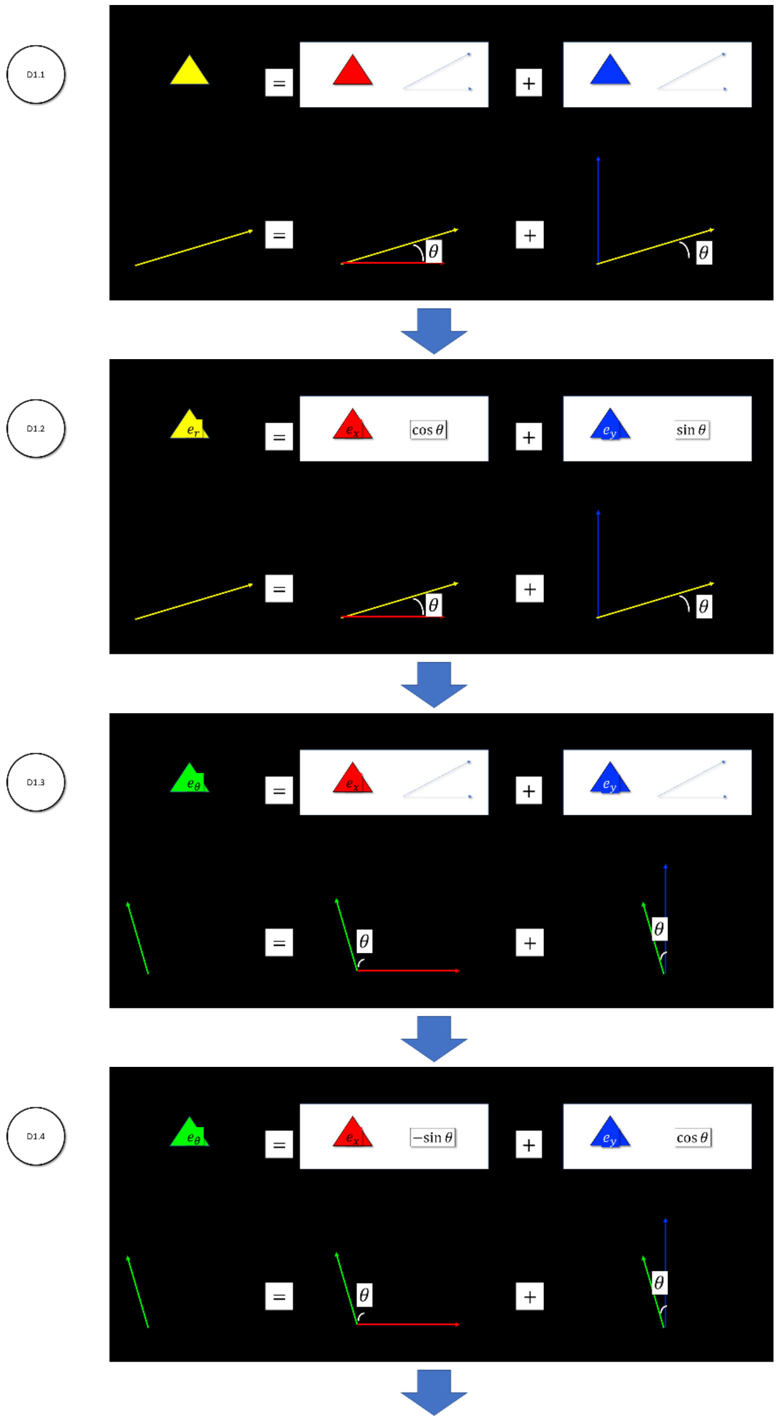
D1.3. In the third step, the angular component of ***b*** is transformed in the same manner as the radial component in D1.1.

D1.4. The name of each component and trigonometric function is revealed in the same manner as the radial component in D1.2.

D1.5. The transformations created for the radial and angular component vectors in D1.2 are organized because they will be used in substitutions in the decoding phases.

D1.6. The transformations created in D1.4 for the x- and y-component vectors are organized.

This storyboard representation of a sample engineering problem is a preliminary look at the potential use of color, shape, and vocabulary in reducing cognitive load. The use of color and shape create contrast while reinforcing mathematical relationships that may help students with math and visual disabilities by avoiding the use of confusing math notation at the beginning of the problem. Furthermore, by pairing the equation of a vector with the representation of the geometry, the cognitive load of the problem is decreased because students do not have to allocate cognitive resources required when switching between the mathematical manipulation of the problem-solving process and an understanding of what that manipulation means geometrically. The goal of reducing cognitive load with PDM is to allow students to focus on the meaning of the math and deepen their understanding of the topic. The development of the method for more topics in engineering is a component of future work planned by researchers.


D1.1
D1.2
e_r = e_x cos θ + e_y sin θ
D1.3
D1.4
e_θ = e_x −sin θ + e_y cos θ
θ

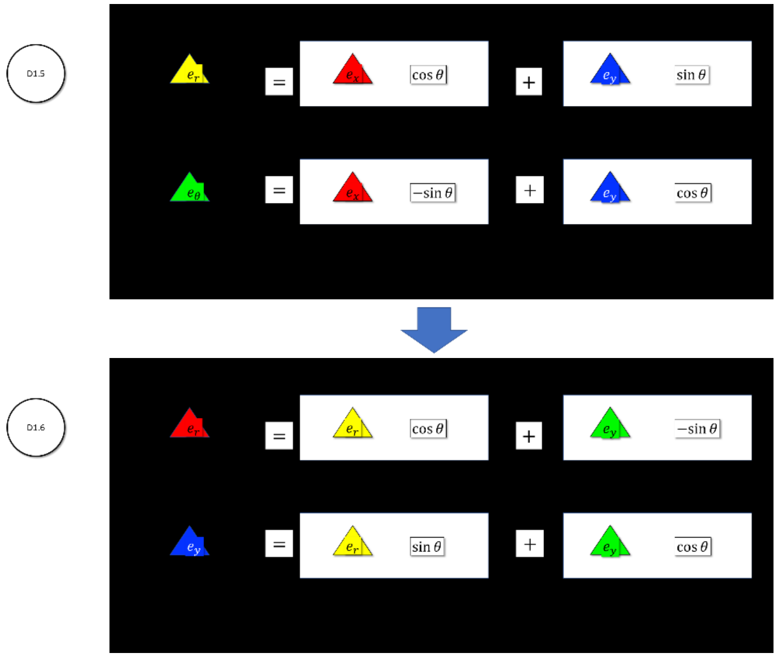


Figure 6. The first phase of the decoding process.

Conclusion

The complexity of mathematics problems hinders the performance of many students, and the research team is considering ways the PDM method may be of benefit to students both with and without disabilities. According to Moon, Todd, et al. [17], 10% of students in STEM majors report having disabilities. However, the survey results in Figure 1 indicate that many students without a diagnosed disability still experience difficulties related to vision, reading, and cognition. Our ongoing studies are focused on the effect that PDM has on all students in college algebra courses. Next steps in development include the creation of PDM modules for many topics in college algebra that will be implemented in tutoring environments. The modules will function as educational support scaffolds to improve access to course content for students with and without disabilities. On the data collection side, the goal of the research team is to analyze quantitative data collected and statistically test differences in performance between students who did and did not receive PDM lessons on topics in college algebra to test the efficacy of PDM. The research team will also begin collecting qualitative data in the form of focus groups and interviews with students and teachers who experience PDM in both pre-college and college settings. Participants will include students and faculty at four higher education environments in the state of Alabama as well as secondary students and their teachers at schools for the blind in three different states. The goal of these interviews will be to gain a detailed understanding of the experiences students had prior to and after their introduction to PDM.

The research team anticipates making significant contributions to the understanding of the relationship between mathematics and the theories of chunking, working memory, and cognitive load. The approach taken in executing the research will also be grounded in well-established educational and design theories. Both Universal Design for Learning (UDL) and User Centered

Design (UCD) provide philosophical frameworks that promote equity in education and inclusive learning environments.

UDL is a design philosophy for curriculum based on the primary goal of creating equitable learning environments for all learners [18]. The UDL philosophy prioritizes the removal of barriers to learning from the inception of curriculum development. PDM follows the UDL philosophy because inclusive learning and accessibility have been guiding philosophies throughout the development of the method. One of the three key principles of UDL is providing multiple means of representation in instruction. A significant goal for the research team is the development of a flexible software application for PDM that incorporates multiple modes of access such as refreshable braille display or audio format for learners who are visually impaired: video with closed captioning for students who are deaf; and simplified, high-contrast displays for students with dyslexia or dyscalculia. When multiple means of representation are considered from the outset and built into the format from the beginning, information is accessible to many more students and education environments are more inclusive. Thus, classroom instruction does not intensify inequities created by student variations like ability status.

UCD is a design philosophy in which the motivations, abilities, and constraints of the users primarily drives the design of a technology [18]. To implement the UCD philosophy, designers understand the end goals of the user and personalize the design of the product around the constraints and allowances of the user's abilities. For example, translation of PDM into software applications and digital learning environments will involve direct collaboration with students who will use the technology during every phase of research and development.

Essential to the expansion of the research is continued adherence to these principles, which have guided the research from its inception. The student with whom and for whom the PDM method was originally created is a highly respected member of the research team. His tenacious pursuit of mathematics learning drove the development of PDM. Expansion of the method continues today, and the students who partner with the research team bring wisdom and insight from their own lived experiences that are essential to the ongoing development and implementation of the method.

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Appendix

Below is a full depiction of the storyboard processes that were summarized in the body of the paper. The application of PDM involves both the encoding and decoding processes. In the encoding phase, the student builds the problem with graphic models. The models consist of colored shapes and spatial representations that are consistent with structures found in the problem. After encoding, the student begins the decoding process. In decoding, the student transforms the problem using the same graphic models. The steps below depict preliminary work that will evolve as evidence-based research informs the ongoing development of the method.

For the sake of clarity, the problem is restated below:

> *2.5 In solving a problem, one person uses cylindrical polar coordinates whereas another uses Cartesian coordinates. To check that their answers are identical, they need to examine the relationship between the Cartesian and cylindrical polar components of a certain vector, say* $\boldsymbol{b} = b_r \boldsymbol{e_r} + b_\theta \boldsymbol{e_\theta}$*. To this end, show that*
>
> $$b_x = \boldsymbol{b} \cdot \boldsymbol{E_x} = b_r \cos\theta - b_\theta \sin\theta, \qquad b_y = \boldsymbol{b} \cdot \boldsymbol{E_y} = b_r \sin\theta + b_\theta \cos\theta.$$

Table of Graphic Models

The graphic models required to both encode and decode the problem are summarized in Figure 5. The elements are displayed in tabular form and function as an advanced organizer as students begin encoding the problem.

There are three categories of graphic models for this problem:

- Geometry models are shapes used in the spatial representation of the problem.
- Variable models are shapes used to abstractly represent variables in simplified equations.
- Tool models are shapes that represent mathematical operations used to simplify problems in the decoding process, such as combining like terms or making substitutions.

The encoding process includes the selection of colors to represent specific sub-structures of the problem. The graphic models are then correctly and consistently color-coded throughout the remaining encoding and decoding processes.

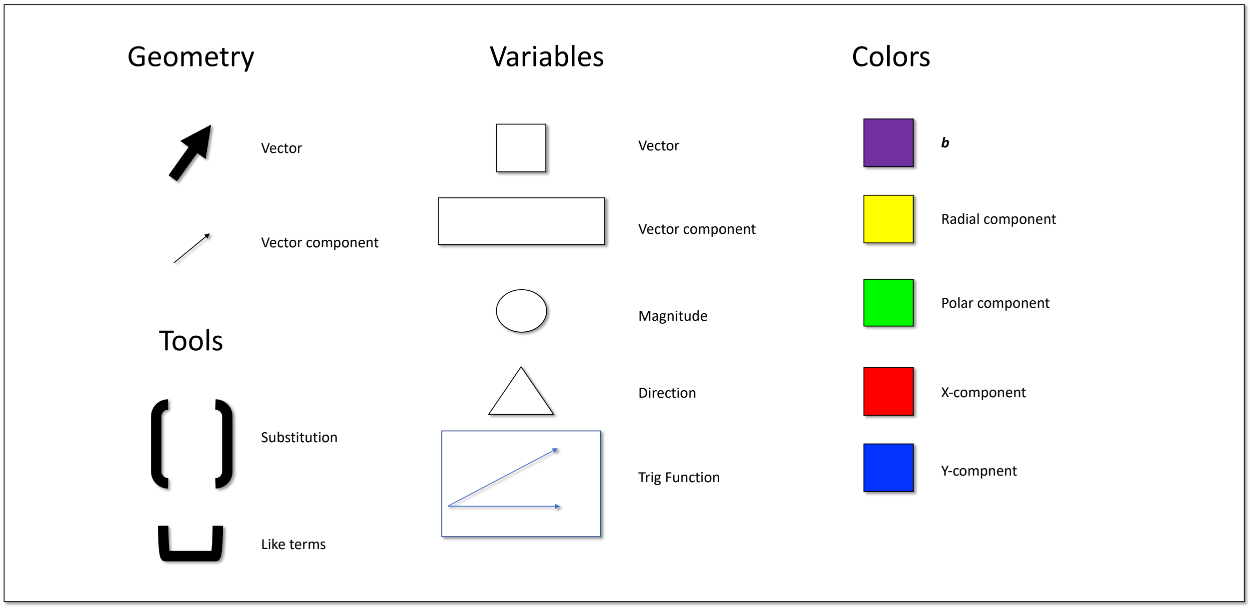


Figure 7. The graphic models and colors used in the encoding and decoding processes.

Spatial Representation

The representation of vectors as the sum of component vectors is shown below in Figures 8-10. Because the problem uses two different coordinate systems, vector $\boldsymbol{b}$ is defined with respect to each system. The figures allow students who may be unfamiliar with these systems to clearly see how they differ spatially. Furthermore, overlapping the different coordinate systems makes clear that vector $\boldsymbol{b}$ still has the same overall magnitude and direction in either system; however, its component vectors differ. Hence, the student can see that vector $\boldsymbol{b}$ is equivalent in either system, helping the student to conceptualize the solution to the problem. In the figures, the scalar and vector components of $\boldsymbol{b}$ use the same color coding to reinforce the relationship between each vector and the terms in the equations used to represent them.

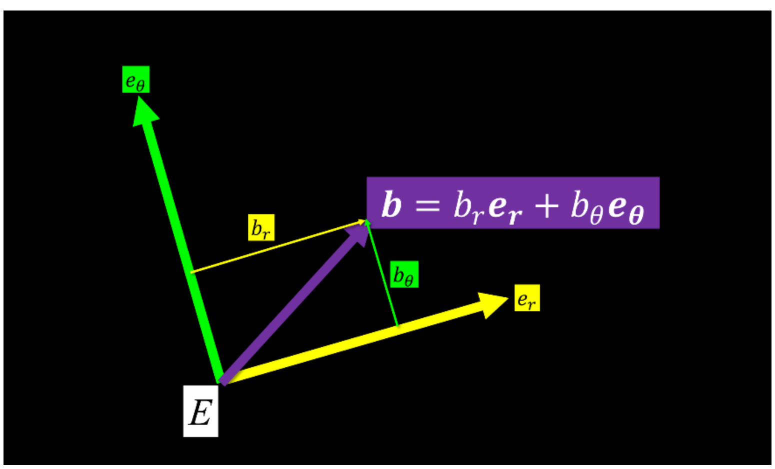


Figure 8. Vector $\boldsymbol{b}$ shown in a cylindrical polar coordinate system.

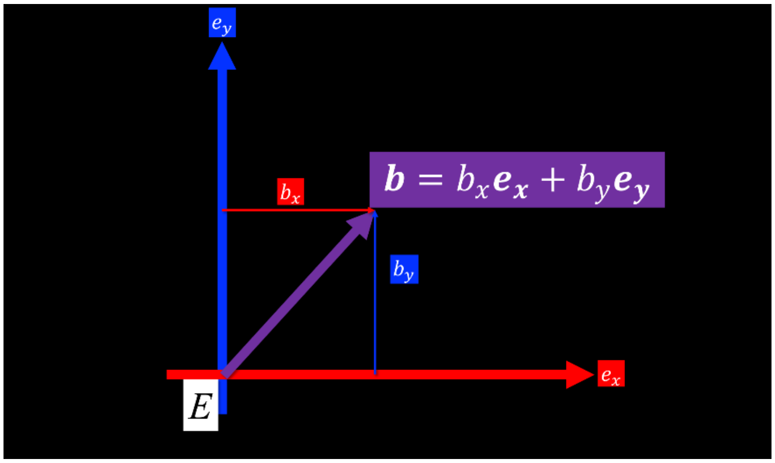


Figure 9. Vector ***b*** shown in a Cartesian coordinate system.

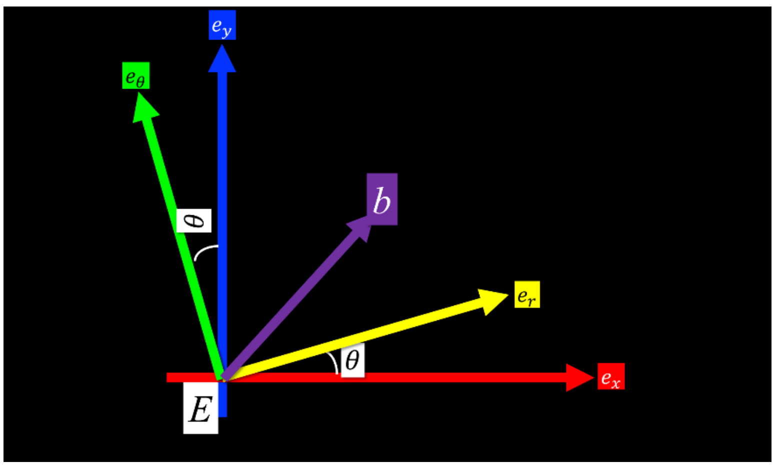


Figure 10. The component vectors of ***b*** from each coordinate system overlapping to highlight the equivalence of ***b*** in each system.

Encoding

First Stage of the Encoding Process:

E1.1. The equation for vector ***b*** is comprised of three rectangles representing the three terms of the equation. Vector ***b*** is represented in both its equation form and its geometric form. In its equation form, vector ***b*** is a purple square labeled with a bold, italicized letter "b." It is set equal to the radial and angular components of the vector that are colored rectangles. The yellow rectangle is labeled "radial component" and the green rectangle is labeled "angular component". The geometric form of the equation is lined up directly below the original equation and uses arrows to represent each term in the vector. Arrows are the same color as their analogous terms directly above them.

E1.2. The radial and angular components are each shown to contain a circle and a triangle. Consistency with color is maintained; yellow is again used for radial and green is again used for angular. The circles represent the scalar magnitude of each component and the triangles represent the direction of each component.

E1.3. The component terms with subscripts of vector ***b*** are then revealed to represent the full complexity of the vector in polar coordinate form.

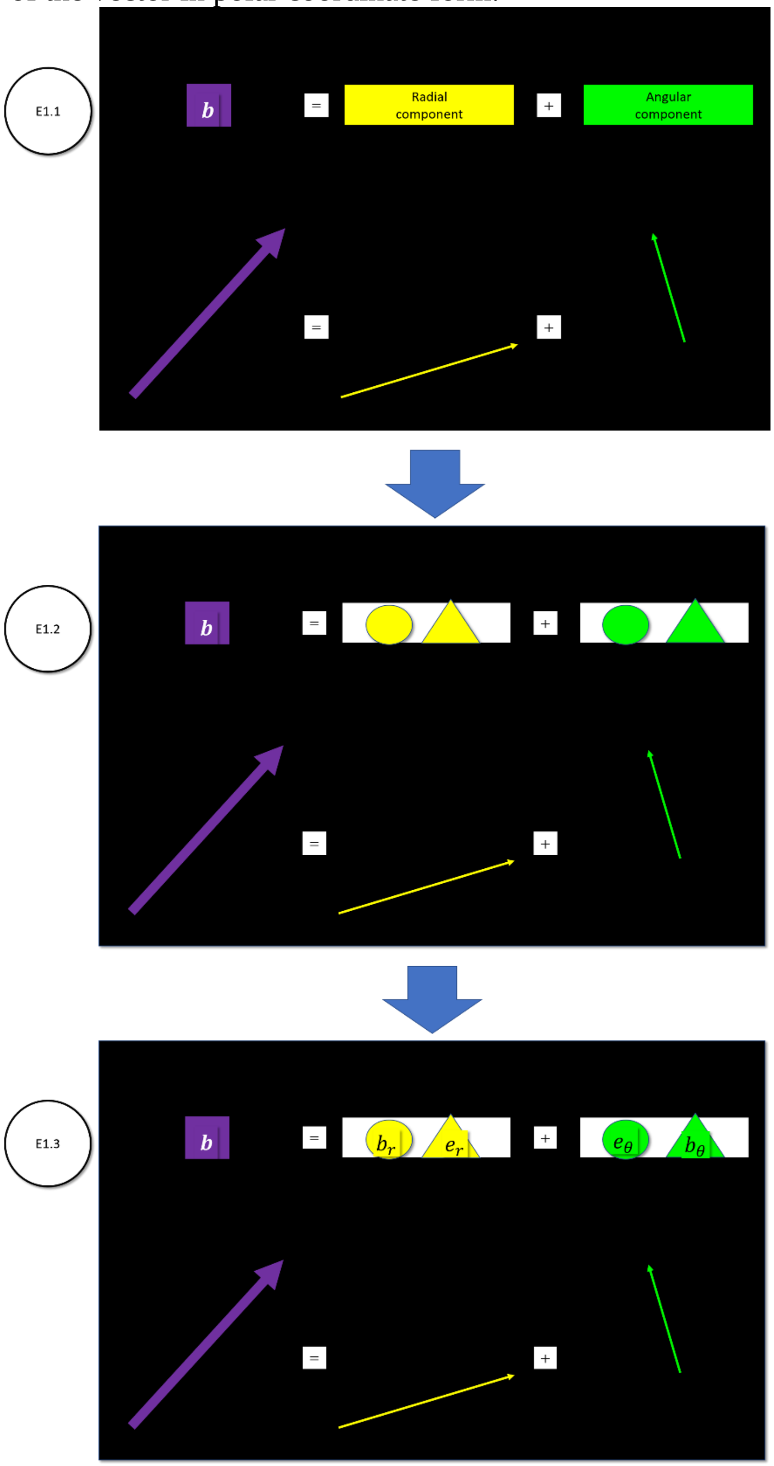


Figure 11. The first stage of the encoding process applied to the dynamics problem solved using PDM (same as Figure 2).

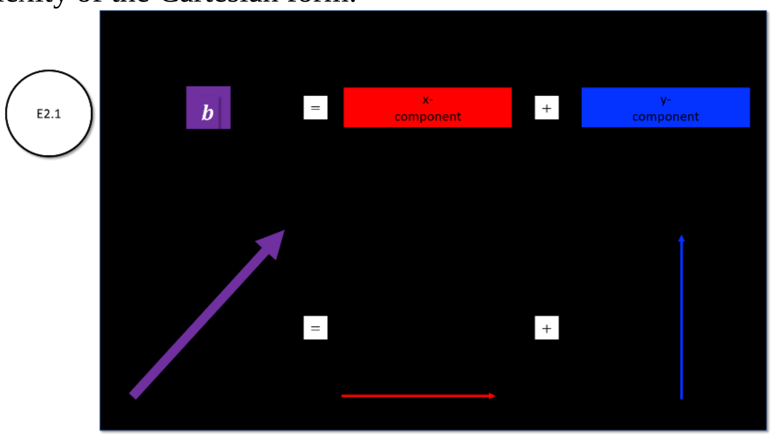
Second Stage of the Encoding Process:

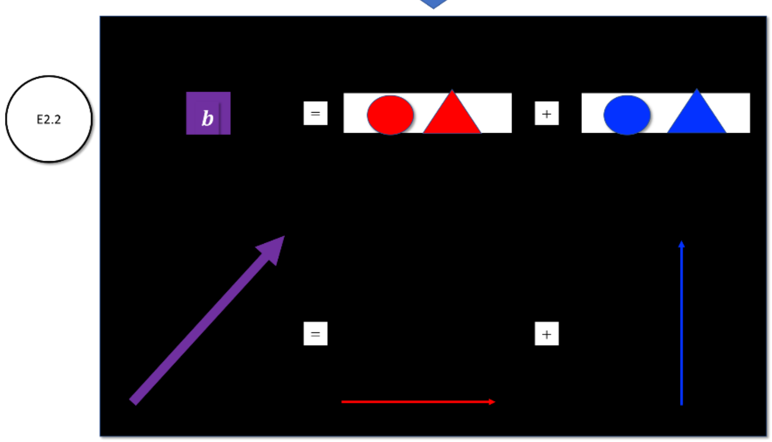
Consistent with Stage 1, Stage 2 applies a similar encoding process where the components of vector ***b*** are defined for the Cartesian coordinate system:

E2.1. A purple square is used for vector ***b***, and red and blue rectangles are used to represent the x- and y-component vectors, respectively. The geometric representation of vector ***b*** is shown below the graphic model representation.

E2.2. The x- and y-components are each shown to contain a circle and a triangle shaded as the same colors as corresponding structures in 2.1. The circles represent the scalar magnitude of each component and the triangles represent the direction of each component.

E2.3. The Cartesian component terms with subscripts of vector ***b*** are then revealed to represent the full complexity of the Cartesian form.





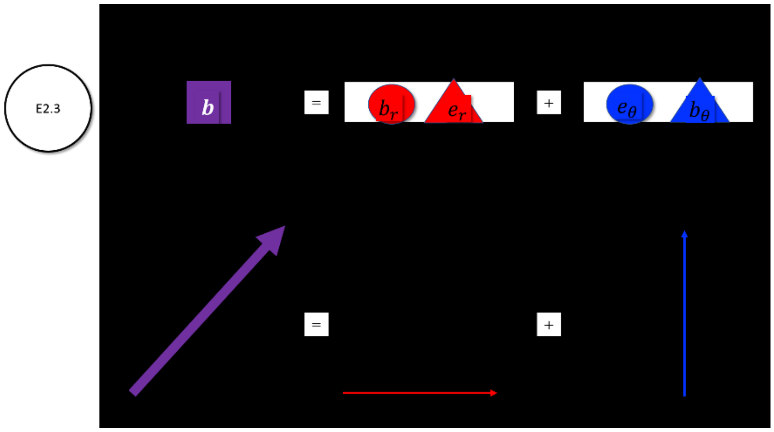


Figure 12. The second stage of the encoding process.

Decoding

First Stage of the Decoding Process:

To solve the problem, the cylindrical polar coordinate components must be transformed into a system of Cartesian coordinate components using trigonometric functions. The same shapes and colors used to represent previous terms above are reused to maintain continuity in a six-step process. The equation forms of each step are also paired with the geometric representation, again to make clear the spatial relationship inherent to the vector transformation:

D1.1. The transformation for the radial direction unit vectors of ***b*** into x- and y-unit direction vectors is represented by an equation with three terms. The trigonometric functions are represented by two vectors intersecting at their tails.

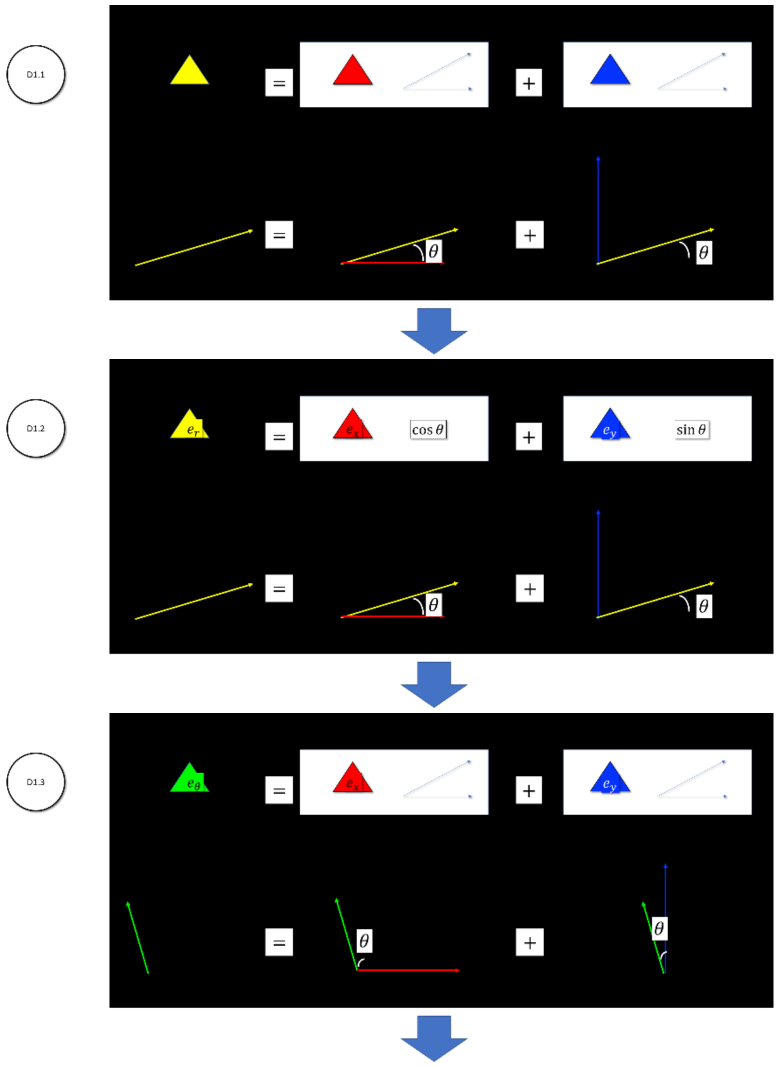
D1.2. The name for each component is then revealed as well as the specific trigonometric function needed.

D1.3. In the third step, the angular component of ***b*** is transformed in the same manner as the radial component in 1.1.

D1.4. The name of each component and trigonometric function is revealed in the same manner as the radial component in 1.2.

D1.5. The transformations created for the radial and angular component vectors in 1.2 are organized to make substitutions later in the decoding phases easier to complete.

D1.6. The transformations created for the x- and y-component vectors in 1.4 are organized.


D1.1
=
+
θ
D1.2
e_r
e_x
cos θ
e_y
sin θ
D1.3
e_θ

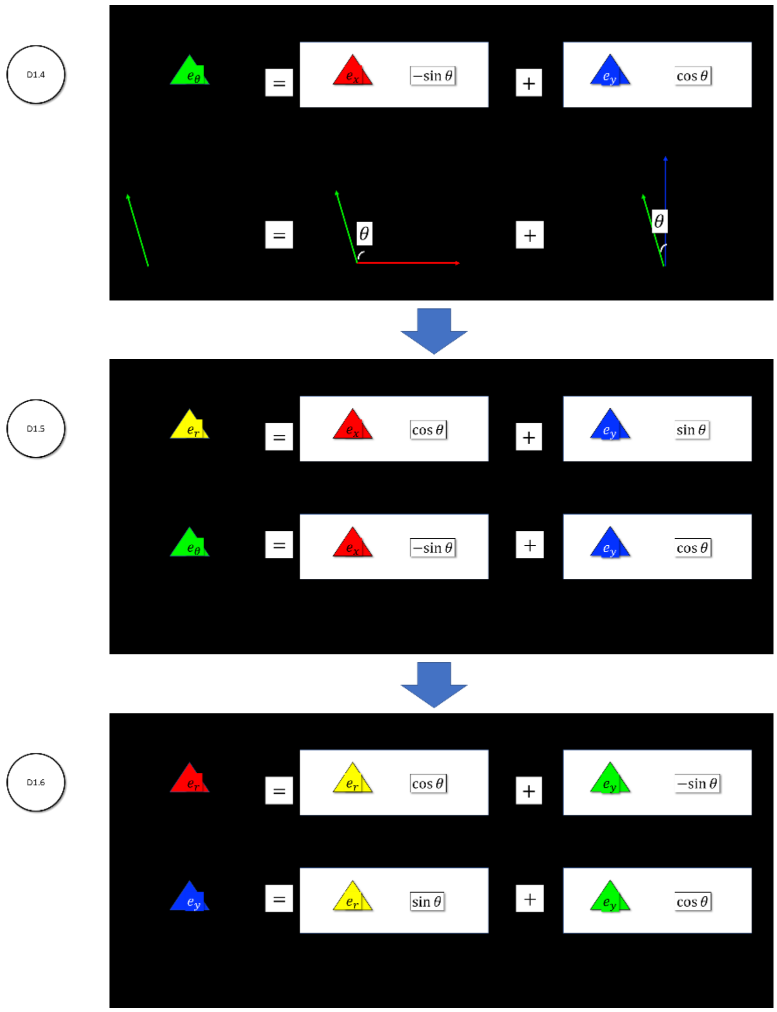


Figure 13. Phase 1 of the decoding process.

Second Stage of the Decoding Process:

In the final stage of decoding, the student solves the problem by using a nine-step series of substitutions and simplifications:

D2.1. The student uses two pairs of the substitution tool to bracket each occurrence of the radial and angular directional terms. The equations for the transformation of the radial and angular directions are below the equation for vector ***b*** to reinforce the connection between the direction terms in the equation and the transformations of the direction terms. The color of the substitution tool matches the color of the term to be substituted to reinforce the relationship between the terms and the resulting substitution.

D2.2. The substitution previewed in 2.1 is carried out and the substituted terms are bracketed with the color corresponding to the term that has been substituted.

D2.3. The student uses two pairs of the substitution tool again to bracket the x- and y-directional terms. Each bracket is color coded to maintain consistency with the color of each variable being substituted. The red pairs of brackets correspond with the x-directional component. The blue pairs correspond with the y-directional component.

D2.4. The substitution previewed in 2.3 is carried out and the substituted terms are bracketed with the color corresponding to the term that has been substituted.

D2.5. The trigonometric functions are multiplied through.

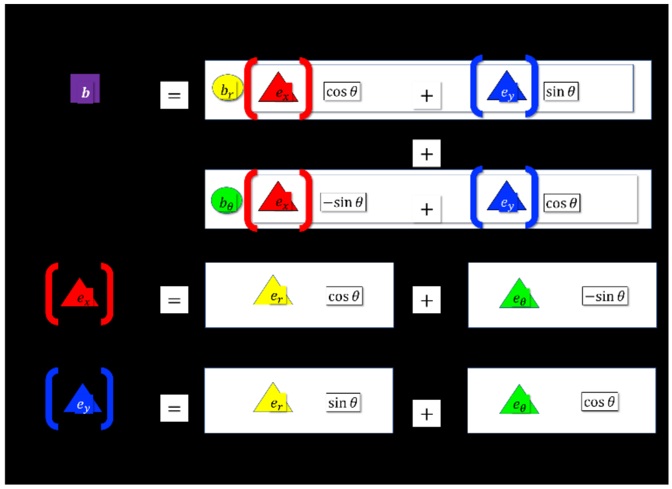
D2.6. Like terms are combined.

D2.7. The resulting simplified equation is shown.

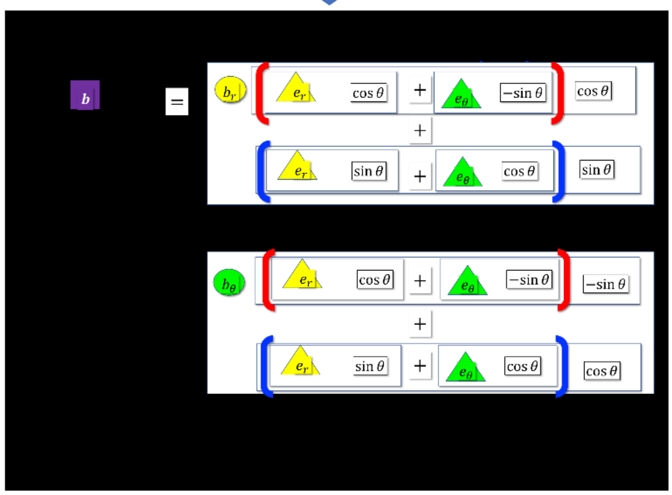
D2.8. The radial and angular components are factored to make clear a trigonometric identity crucial to the problem's solution. The identity is bracketed, along with the corresponding locations in the above equation where the substitution of the identity will be made to make the process of the substitution clearly understandable.

D2.9. The resulting simplified equation results in the equation for ***b*** in radial components, proving that ***b*** is equivalent in either coordinate system.

D2.3


b
=
b_r e_x $\cos\theta$ + e_y $\sin\theta$
+
b_θ e_x $-\sin\theta$ + e_y $\cos\theta$
e_x = e_r $\cos\theta$ + e_θ $-\sin\theta$
e_y = e_r $\sin\theta$ + e_θ $\cos\theta$

D2.4


b
=
b_r e_r $\cos\theta$ + e_θ $-\sin\theta$ $\cos\theta$
+
e_r $\sin\theta$ + e_θ $\cos\theta$ $\sin\theta$
b_θ e_r $\cos\theta$ + e_θ $-\sin\theta$ $-\sin\theta$
+
e_r $\sin\theta$ + e_θ $\cos\theta$ $\cos\theta$

D2.5

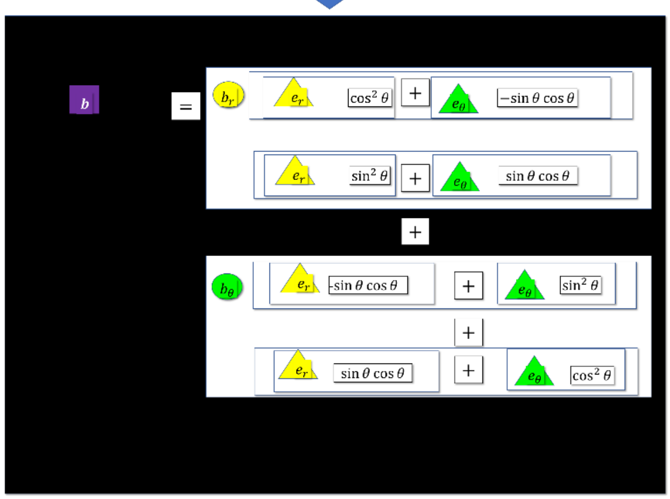

b
=
b_r e_r $\cos^2\theta$ + e_θ $-\sin\theta\cos\theta$
e_r $\sin^2\theta$ + e_θ $\sin\theta\cos\theta$
+
b_θ e_r $-\sin\theta\cos\theta$ + e_θ $\sin^2\theta$
+
e_r $\sin\theta\cos\theta$ + e_θ $\cos^2\theta$

Figure 14. The final decoding process resulting in the solution of the problem.