A New Method for Teaching The Fourbar Linkage and its Application to Other Linkages

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A New Method for Teaching the Fourbar Linkage to Engineering Students

Abstract

The fourbar linkage is one of the first mechanisms that a student encounters in a machine kinematics or mechanism design course and teaching the position analysis of the fourbar has always presented a challenge to instructors. Position analysis of the fourbar linkage has a long history, dating from the 1800s to the present day. Here position analysis is taken to mean 1) finding the two remaining unknown angles on the linkage with an input angle given and 2) finding the path of a point on the linkage once all angles are known. The efficiency of position analysis has taken on increasing importance in recent years with the widespread use of path optimization software for robotic and mechanism design applications.

Kinematicians have developed a variety of methods for conducting position analysis, but the solutions presented in the literature fall into two general families:

1. The angle between the coupler and the rocker is found using the law of cosines. Once this is known, the coupler and rocker angles are found using some combination of the laws of sines and cosines.
2. A vector loop equation is written around the linkage, and then half-angle tangent identities are used to solve for the two unknown angles.

Two widely-used mechanical design textbooks use method 2, whose derivation is lengthy and whose final results permit no simple geometric interpretation. Method 1 has a much simpler derivation but is difficult to implement in software owing to a lack of four-quadrant functions for sine and cosine.

With this in mind, we have developed a more efficient method for obtaining the position solution for the fourbar linkage that is well-suited to educational settings as well as for design optimization: the projection method. Because the final formulas have an elegant geometric interpretation, we have found that this method is easier for mechanical engineering students to understand and could therefore become a new standard method for mechanical design textbooks. In addition, the final position formula uses the tangent function, which has widely-available four-quadrant implementations. The projection method is easily extended to other common linkages, including the inverted slider-crank, the geared fivebar linkage, and four of the five types of single degree-of-freedom sixbar linkages. This method has been used to develop an educational website, www.mechdes.net, that contains simulations of several common linkages and mechanisms. This paper presents a comparison of the two traditional methods and the projection method, and pseudocode algorithms for each method are given at the end.

Keywords

Fourbar linkage, fourbar mechanism, linkage optimization, position analysis

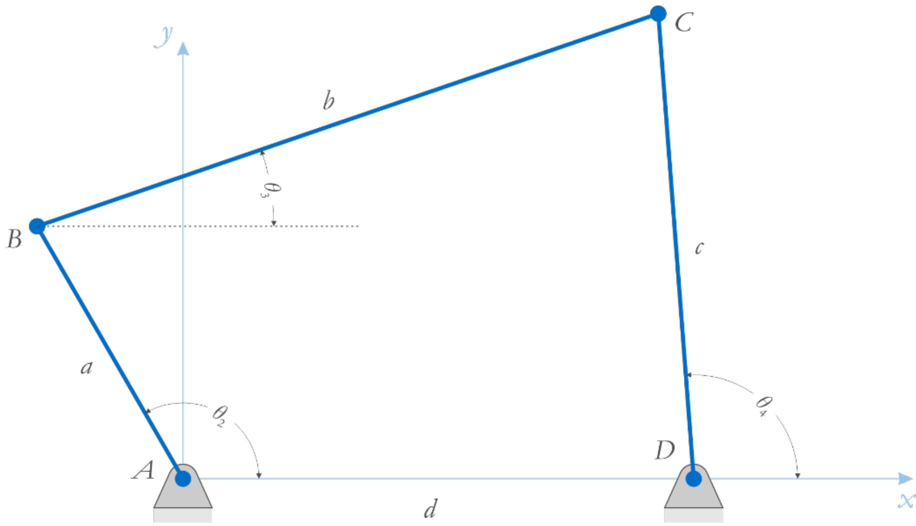

Figure 1: The classic fourbar linkage, with all angles defined from the horizontal. Here we assume that θ_2 is known, and we wish to solve for θ_3 and θ_4.

Introduction

The classic fourbar linkage is shown in Figure 1. In most instances the input to the linkage is the crank angle θ_2. The coupler and rocker angles, θ_3 and θ_4, are usually unknown and the position analysis of the fourbar linkage is generally considered complete once these angles have been found.

Position analysis of fourbar linkages has a long history, from the nineteenth century [1], [2] until the present day [3]. Researchers have developed a variety of methods for conducting position analysis, but the solutions presented in the literature fall into two general families:

> Method 1: The angle between the coupler and the rocker (angle *BCD* in Figure 1) is found using the law of cosines. Once this is known, the coupler and rocker angles are found using some combination of the laws of sines and cosines.
> Method 2: A vector loop equation is written around the linkage, and then half-angle tangent identities are used to solve for the two unknown angles.

Both Norton [4] and Waldron [5] use method 2, whose derivation is lengthy and whose final results permit no simple geometric interpretation. Method 1 has a much simpler derivation and is used by Martin [6], Myszka [7] and Bulatović and Đorđević [8]. The dot product method presented by Wilson and Sadler [9] obtains essentially the same results, but in a more complicated fashion. We describe both methods below in order to compare them with the newly-developed method, which we denote the *projection* method.

This work is a part of ongoing research to establish the most effective method for teaching fourbar linkage analysis to engineering students. Previous related work has established the computational efficiency of deriving the solution for fourbar mechanisms using the projection method [11]. An experiment with undergraduate engineering students is planned to methodically validate its educational efficacy. The goal of this paper is the document the teaching methods that will be employed in the educational study. The study procedures are outlined below.

Table 1. Outline for the experiment to validate the projection method for teaching fourbar mechanism

Task	**Group A**	**Group B**	**Time (min)**	**Description**
Teach half-angle method with partial notes	--------------	--------------	20	General instructions Introduce general fourbar mechanism solution with half-angle method
Control Condition	Problem 1	Problem 2	30	Introduction to the problem Time to solve fourbar problem
Break	--------------	--------------	**10**	-------------------------------
Teach new method	--------------	--------------	20	Introduction to projection method for solving fourbar mechanisms
Experimental Condition	Problem 2	Problem 1	30	Introduction to the problem Time to solve fourbar problem
Participant Feedback	--------------	--------------	10	Collect participant demographics feedback on the study
Total time	--------------	--------------	**120**	**Up to 2 hours**

The validation study will be conducted outside of class where students can enroll voluntarily and receive modest compensation for their time. Students will be subjected to both the methods and the solutions derived for different problems. For problem 1, group A will be the control group and group B will be the experimental. Similarly, for problem 2 group B will be control and group A will be experimental. Thus, we can compare the solutions accuracy as well as the ease of obtaining solutions with both methods for both problems.

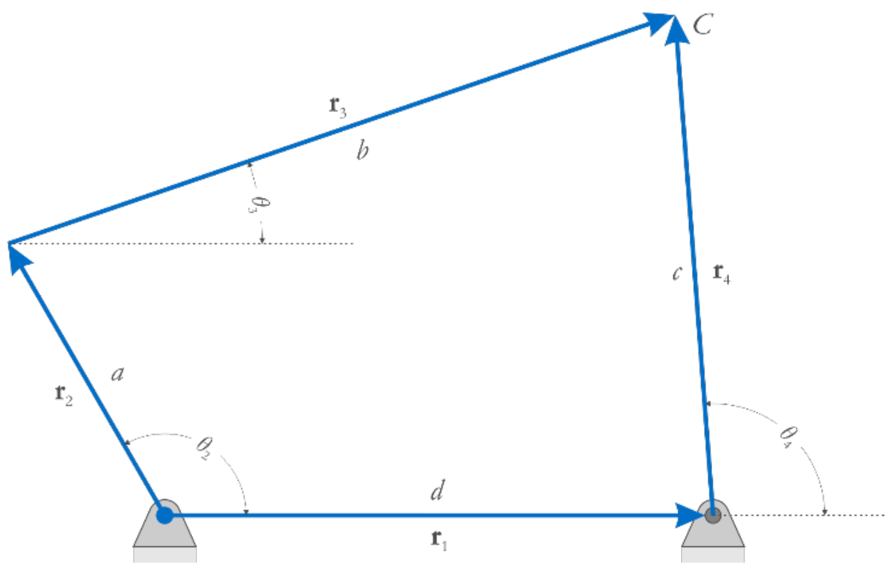

Figure 2: A vector loop can be drawn around the fourbar linkage whose sum is zero.

Half-Angle Method

A common position analysis method is that given by Norton [4], which uses half-angle trigonometric identities to solve for the angles θ_3 and θ_4. The method begins with a vector loop drawn around the linkage, as shown in Figure 2. The vector loop equation can be written

$$\mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4 - \mathbf{r}_1 = \mathbf{0} \qquad (1)$$

Or, in trigonometric form

$$\begin{aligned} a\cos\theta_2 + b\cos\theta_3 - c\cos\theta_4 - d &= 0 \\ a\sin\theta_2 + b\sin\theta_3 - c\sin\theta_4 &= 0 \end{aligned} \tag{2}$$

We first solve for θ_4 by eliminating θ_3, so place all θ_3 terms on the left side of the equations.

$$\begin{aligned} b\cos\theta_3 &= c\cos\theta_4 - a\cos\theta_2 + d \\ b\sin\theta_3 &= c\sin\theta_4 - a\sin\theta_2 \end{aligned} \tag{3}$$

Take the square of both equations, and add together:

$$b^2 = (c\cos\theta_4 - a\cos\theta_2 + d)^2 + (c\sin\theta_4 - a\sin\theta_2)^2 \tag{4}$$

Expand the squares and simplify:

$$b^2 = c^2 - 2ac(\cos\theta_4\cos\theta_2 + \sin\theta_2\sin\theta_4) + 2cd\cos\theta_4 + a^2 - 2ad\cos\theta_2 + d^2 \tag{5}$$

Now define:

$$K_1 = \frac{d}{a} \qquad K_2 = \frac{d}{c} \qquad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

Then Equation (5) becomes

$$\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4 = K_3 + K_1\cos\theta_4 - K_2\cos\theta_2 \tag{6}$$

We can use the half-angle identities to reduce Equation (6) to a single trigonometric function.

$$\sin\theta_4 = \frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} \qquad \cos\theta_4 = \frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} \tag{7}$$

Substitute these into (6) to give

$$(\cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3)\tan^2\left(\frac{\theta_4}{2}\right) - 2\sin\theta_2\tan\left(\frac{\theta_4}{2}\right) + K_3 + K_1 - (K_2+1)\cos\theta_2 \tag{8}$$

At this point, of course, the ordinary Mechanical Engineering student has thrown up his or her hands in frustration and has stopped paying attention. There would appear to be no tangible geometric interpretations to the *K* terms, and the half-angle trigonometric identities are arcane, at best. But, to continue, let us define

$$\begin{aligned} A &= \cos\theta_2 - K_1 - K_2\cos\theta_2 + K_3 \\ B &= -2\sin\theta_2 \\ C &= K_1 - (K_2+1)\cos\theta_2 + K_3 \end{aligned} \tag{9}$$

Then we have a familiar-looking quadratic equation

$$A\tan^2\left(\frac{\theta_4}{2}\right) + B\tan\left(\frac{\theta_4}{2}\right) + C = 0 \tag{10}$$

that we can easily solve for $\theta_4/2$. The half-angle method is relatively straightforward to implement in a spreadsheet or MATLAB® but suffers from the significant disadvantage of having no clear geometric interpretation for any of the intermediate solution variables. In addition, there is no method for the student to verify the calculations for *A*, *B*, *C* or the *K* values independently. Since one of our primary goals as educators is to ensure that our students learn the importance of verifying calculations, the half-angle method is difficult to justify in an educational setting.

The Method of Bulatović and Đorđević

The method of Bulatović and Đorđević [8] does not rely on half-angle identities and allows a simple geometric interpretation. The coordinates of point *B* in Figure 3 are

$$\begin{aligned} x_B &= a\cos\theta_2 \\ y_B &= a\sin\theta_2 \end{aligned} \tag{11}$$

Let us define the *prime diagonal*, *f*, to be the line between *B* and *D* [9]. The length of *f* is given by

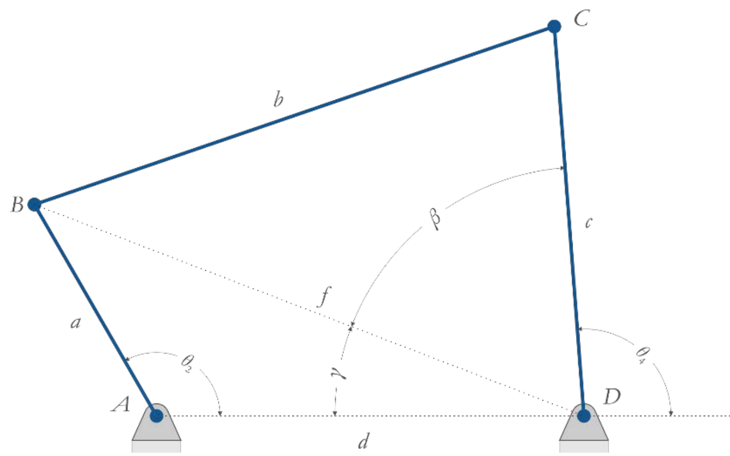

Figure 3: Angles used in the method of Bulatović and Đorđević

$$f^2 = (d - x_B)^2 + y_B^2 \tag{12}$$

The law of cosines gives the angle β

$$\cos\beta = \frac{f^2 + c^2 - b^2}{2fc} \tag{13}$$

And the angle γ is given by

$$\tan\gamma = \frac{y_B}{d - x_B} \tag{14}$$

From Equations (13) and (14), it is simple to deduce the angle θ_4.

$$\theta_4 = \pi - \gamma - \beta \tag{15}$$

Once θ_4 is known, we may solve for the coordinates of point *C*.

$$\begin{aligned} x_C &= c\cos\theta_4 + d \\ y_C &= c\sin\theta_4 \end{aligned} \tag{16}$$

The angle θ_3 is found using the inverse tangent.

$$\tan\theta_3 = \frac{y_C - y_B}{x_C - x_B} \tag{17}$$

Thus, the method of Bulatović and Đorđević produces the angles θ_3 and θ_4 in a few steps, and each step is easy to justify from a geometric perspective. The primary disadvantage of the method is that it requires the calculation of a square root and several trigonometric functions within the solution loop.

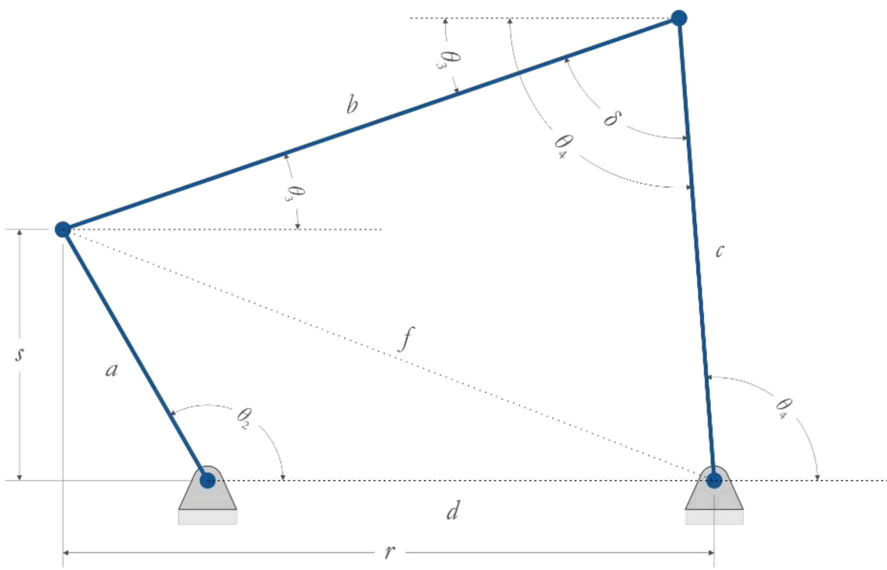

Figure 4: Geometry used in the projection method, with the prime diagonal shown as f.

The Method of Projections

Let us take a closer look at the prime diagonal as shown in Figure 4. First, using the Pythagorean Theorem, we note that

$$f^2 = r^2 + s^2 \tag{18}$$

Where

$$r = d - a\cos\theta_2$$
$$s = a\sin\theta_2 \tag{19}$$

Thus,

$$f^2 = a^2 + d^2 - 2ad\cos\theta_2 \tag{20}$$

The reader may recognize the expression above as a restatement of the Law of Cosines. Now define the angle opposite θ_2 as δ. We can also use the Law of Cosines to write

$$f^2 = b^2 + c^2 - 2bc\cos\delta \tag{21}$$

or, solving for δ, we have

$$\cos\delta = \frac{b^2 + c^2 - f^2}{2bc} \tag{22}$$

The angle δ is opposite θ_2 in the quadrilateral. In Figure 4, we can also see that

$$\delta = \theta_4 - \theta_3 \tag{23}$$

which means that we need only solve for θ_3, since Equation (23) can be used to find θ_4.

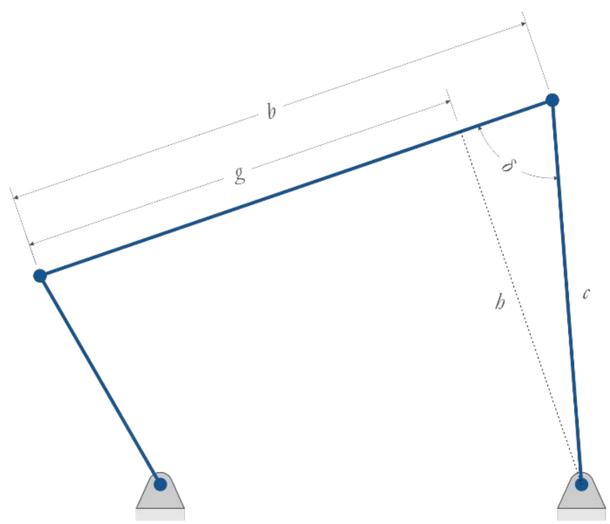

Figure 5: The lengths g and h can be found once the angle δ is known.

Now that we know the angle δ, we can use it to calculate a few more interesting quantities. Project a perpendicular line from the coupler to the rocker pin, as shown in Figure 5. Define the new lengths

$$g = b - c\cos\delta \tag{24}$$

$$h = c \sin \delta$$

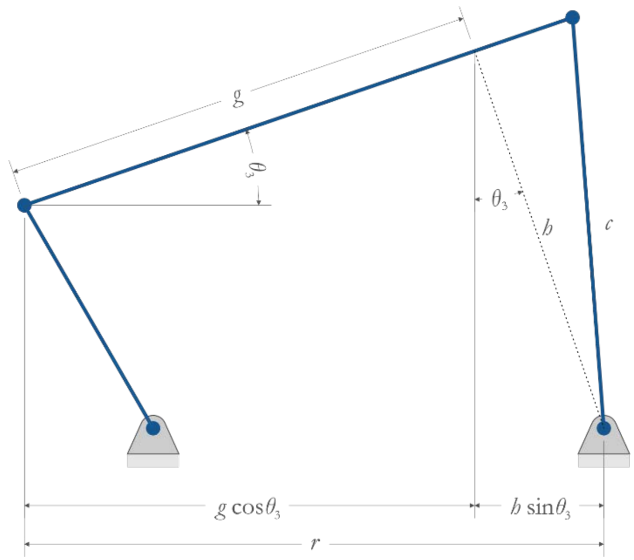

Figure 6: The dimensions g and h can be related to r through the angle θ_3.

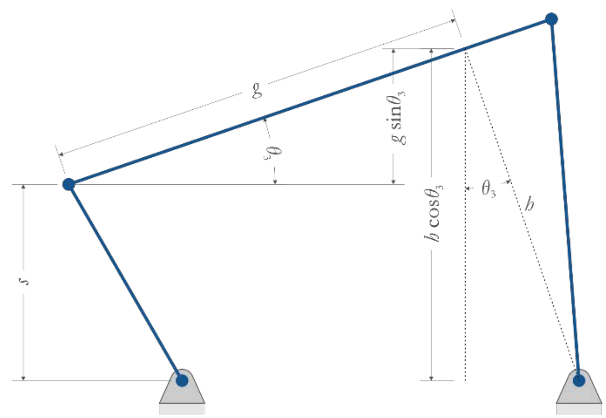

Figure 7: The dimensions g and h can be related to s through the angle θ_3.

Next, going back to the variables *r* and *s* defined earlier, we can write

$$\begin{aligned} r &= g\cos\theta_3 + h\sin\theta_3 \\ s &= h\cos\theta_3 - g\sin\theta_3 \end{aligned} \tag{25}$$

as shown in Figure 6 and Figure 7. We now have two equations with one unknown, θ_3. Each equation is transcendental and difficult to solve on its own. Therefore, we will employ a few tricks to isolate θ_3. First, divide both equations by cos θ_3

$$\begin{aligned} \frac{r}{\cos\theta_3} &= g + h\tan\theta_3 \\ \frac{s}{\cos\theta_3} &= h - g\tan\theta_3 \end{aligned} \tag{26}$$

Then, solve both for $\cos\theta_3$

$$\cos\theta_3 = \frac{r}{g + h\tan\theta_3}$$
$$\cos\theta_3 = \frac{s}{h - g\tan\theta_3} \tag{27}$$

Set the two equations equal to each other

$$\frac{r}{g + h\tan\theta_3} = \frac{s}{h - g\tan\theta_3} \tag{28}$$

And finally, solve for $\tan\theta_3$.

$$\tan\theta_3 = \frac{hr - gs}{gr + hs} \tag{29}$$

Once we have calculated θ_3, we can use (23) to calculate θ_4. Thus, we have achieved our goal of finding the two unknown angles of the fourbar linkage. This method has the added feature of employing the tangent function (as opposed to sine or cosine). When we solve these equations using MATLAB® or Excel, we can use the **atan2** function to solve for θ_3 in any quadrant.

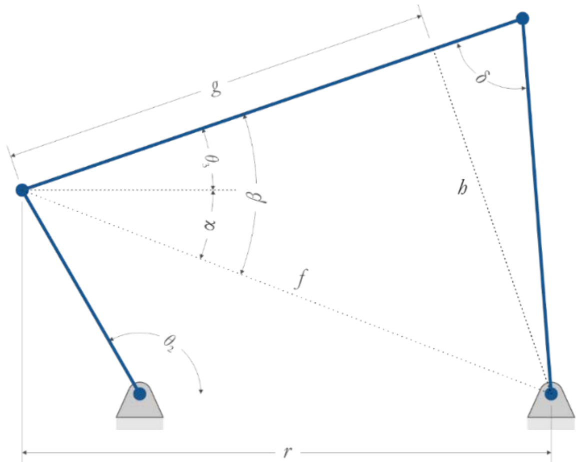

Figure 8: The angle θ_3 can also be found using an angle sum trigonometric identity.

A Digression into Trigonometric Identities

Let us approach the tangent formula given in Equation (29) from a different angle, as it were. If we examine a table of trigonometric identities, we will usually find a tangent sum formula

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v} \tag{30}$$

Examining Figure 8, we see that

$$\theta_3 = \beta - \alpha \tag{31}$$

so that

$$\tan\theta_3 = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} \tag{32}$$

where

$$\tan\alpha = \frac{s}{r} \qquad\qquad \tan\beta = \frac{h}{g} \tag{33}$$

Substituting these into Equation (32) gives

$$\tan\theta_3 = \frac{\dfrac{h}{g} - \dfrac{s}{r}}{1 + \dfrac{h}{g}\cdot\dfrac{s}{r}} = \frac{hr - gs}{gr + hs} \tag{34}$$

as before. There is more than one way to arrive at our position formula!

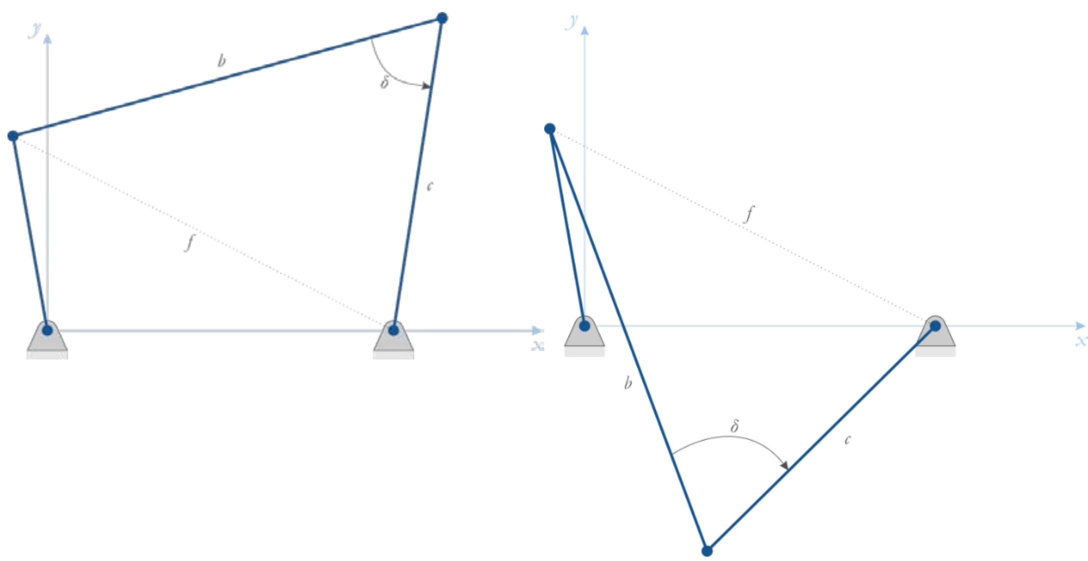

Figure 9: Open and crossed configurations of a fourbar linkage.

Open and Crossed Configurations of the Fourbar

Figure 9 shows a typical fourbar linkage in its "open" and "crossed" configurations. We have used the open configuration to define the sense of the angles in our formulas. For example, the angle δ was defined as the angle *from* the coupler *to* the rocker, as shown in Figure 9 at left. Since the direction of this angle is counterclockwise, we consider it to have a positive value. In the crossed configuration, shown at right, the angle from coupler to rocker sweeps in the clockwise direction and is therefore negative. Thus, to switch between the open and crossed configurations in our calculations, we can simply change the sign of δ.

$$\begin{aligned}\delta &= \cos^{-1}\left(\frac{b^2+c^2-f^2}{2bc}\right) \text{ for open} \\ \delta &= -\cos^{-1}\left(\frac{b^2+c^2-f^2}{2bc}\right) \text{ for crossed}\end{aligned} \tag{35}$$

This operation is mathematically valid because the cosine function gives the same result for positive and negative angles

$$\cos(\delta) = \cos(-\delta) \tag{36}$$

All of the remaining formulas for θ_3 and θ_4 are the same as before.

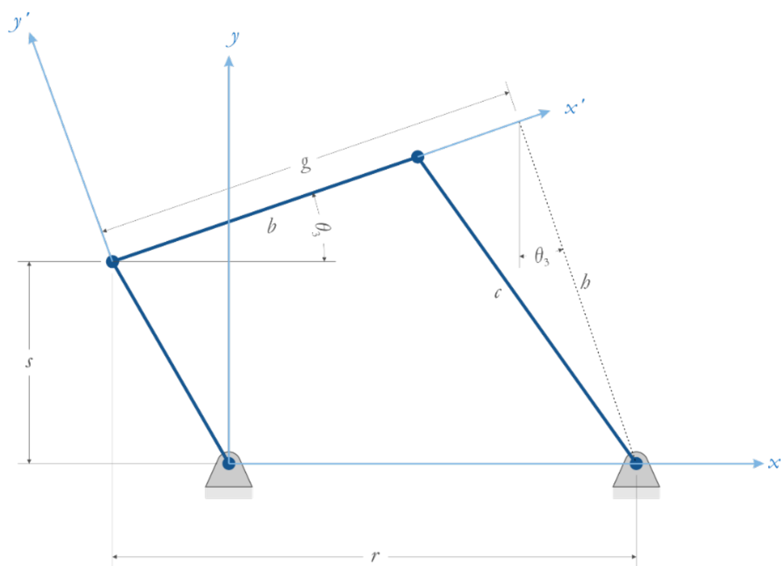

Figure 10: The projection method can be interpreted as a coordinate transformation between the xy and x'y' systems.

The Projection Method as Coordinate Transformation

Equations (25) may be rewritten

$$\begin{aligned}h\sin\theta_3 + g\cos\theta_3 &= r \\ h\cos\theta_3 - g\sin\theta_3 &= s\end{aligned} \tag{37}$$

These are the well-known coordinate transformation equations [10], which transform the components of the vector f between the xy and $x'y'$ coordinate systems, as shown in Figure 10. Thus, the projection method provides an elegant – and for the authors, unanticipated – geometric interpretation. Pseudocode versions of the three algorithms are given in Table 1.

Table 2: Pseudocode version of the three algorithms. Each method assumes that 360 position evaluations are needed.

Half-Angle Method	*Bulatović Method*	*Projection Method*
K1 = d/a K2 = d/c K3 =(a*a-b*b+c*c+d*d)/(2*a*c) K4 = d/b K5 =(c*c-d*d-a*a-b*b)/(2*a*b); for θ2 = 0 to 360 Q = cos(θ2) A = K3 - K1 - (K2 - 1)*Q B = -2*sin(θ2); C = K3 + K1 - (K2 + 1)*Q D = K5 - K1 + (K4 + 1)*Q E = B F = K5 + K1 + (K4 - 1)*Q θ4=2*atan2(-B-sqrt(B*B-4*A*C),2*A) θ3=2*atan2(-E-sqrt(E*E-4*D*F),2*D) end	C = c*c - b*b; for θ2 = 0 to 360 xB = a*cos(θ2)-d yB = a*sin(θ2) f2 = xB*xB + yB*yB f = sqrt(f2) γ = atan2(yB, -xB) β = acos((f2+C)/(2*f*c)) θ4 = π – (γ + β); xC = c*cos(θ4); yC = c*sin(θ4); θ3 = atan2(yC-yB,xC-xB) end	A = b*b + c*c B = 2*b*c for θ2 = 0 to 360 r = d - a*cos(θ2) s = a*sin(θ2) f2 = r*r + s*s δ = acos((A - f2)/B) g = b – c*cos(δ) h = c * sin(δ) θ3 = atan2(h*r - g*s,g*r + h*s) θ4 = θ3 + δ end

Educational Implications of the Projection Method

The authors have a combined total of more than 30 years of experience in teaching Mechanical Design to first-year and third-year mechanical engineering students, with most of that time spent teaching the half-angle method of Norton. We have found it impossible to completely cover the half-angle method in a single class period and have resorted to the "partial notes" method to give students an overview of the method, or to providing students with "black box" routines to allow them to conduct position analysis without doing their own coding. In our experience, there are two primary disadvantages to using the half-angle method in an educational setting:

1. The lack of a clear geometric interpretation prevents students from achieving any kind of engineering intuition as to the meaning of the terms in the equations. This makes checking the intermediate steps of the calculations difficult, at best.
2. The formulas are too complicated for most students (especially first-year students) to implement in software without error, and the lack of a geometric interpretation makes debugging very frustrating for students who are coding novices.

As can be seen in the pseudocode implementation in Table 1, the projection method requires only a few simple lines of code, and we have had great success in having students program their own fourbar linkage solvers in MATLAB and Excel. We are in the process of collecting data on the effectiveness of the projection method in an educational setting. The delay in data collection is for the simple reason that we are reluctant to subject a new group of students in our classes to the half-angle method when the new approach offers such advantages in clarity and ease of implementation. Instead, we chose to collect data in an educational study outside of the class in which students can enroll voluntarily and learn both methods. With such a study we should be also to comparatively assess the effectiveness of the using the projection method in an educational setting. A website with a fourbar simulator can be found at www.benchtophybrid.com/FB/FB_Fourbar.html and simulations of other linkages (e.g. inverted slider-crank) based on the projection method can be found at www.mechdes.net Finally, the projection method has been implemented in a textbook by the authors, *Introduction to Mechanical Design with Computer Applications* (Taylor and Francis, 2018).

Conclusion

We have presented a new method for teaching position analysis on the fourbar linkage. This method allows several elegant geometric interpretations, which make it simple and rewarding to use in an educational setting. We have shown in an earlier paper [11] that the projection method is computationally more efficient and robust than competing methods, which makes it suitable for linkage optimization. We have used this method for three years in our own classrooms and have found that students have little difficulty in following the geometry-based derivation. Our hope is that the projection method will become the standard in teaching position analysis for the fourbar and other linkages (e.g. the inverted slider-crank, geared fivebar, etc.).

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