Determining System Time Constant Through Experimental and Analytical Techniques

Abstract—New experimental procedures are being developed to help students connect course theory to real-world systems. The time constant of a fluid system consisting of a tank, inlet flow, and outlet flow through a valve is determined by the resistance and tank area. The valve resistance is non-linear and depends on the flow rate and tank level. Both the time constant around an operational point and the average time constant through a range of operational points can be determined through experimental and analytical techniques. The time constant of an experimental liquid-level rig is determined by deriving a linear model of the system and experimentally determining the relationship between steady-state outlet flow rate and tank level. These analytical and experimental methods incorporate concepts mechanical engineering students learn in several core courses. Homework assignments and laboratory experiments exploring a physical system serve to engage students and help them make connections between course material spanning the mechanical engineering curriculum.

Keywords—laboratory design; time constant; fluid dynamics; control systems; linearization; valve resistance

I. INTRODUCTION

This project explores several methods of determining the system time constant for a dynamic fluid system through both analytical and experimental techniques. The system, which includes a tank, an inlet flow, and an outlet flow through a valve of unknown resistance, is modeled using approaches taught in the Systems Engineering, Feedback Control Systems and Fluid Mechanics courses. Systems Engineering is a sophomore course which introduces the mathematical modeling and time response analysis of mechanical, electromechanical, fluid, and thermal systems. Feedback Control Systems is taken junior year and is a continuation of Systems Engineering. In Feedback Control Systems students learn various methods of analyzing and controlling linear time-invariant dynamic systems. Students take Fluid Mechanics concurrently with Feedback Control Systems and learn concepts of incompressible viscous flow, including energy considerations and head loss calculations. Faculty and undergraduate research assistants are redesigning the Feedback Control Systems course to include a significant laboratory component.

This project is part of a larger initiative that seeks to offer students a more complete understanding of mechanical engineering concepts by providing hands-on experiences and showing connections to professional practice [1]. Inductive learning techniques, which engage students through hands-on experimental procedures and practical applications are being introduced into the Mechanical Engineering Department’s Feedback Control Systems Course. Since 2008, the class has incorporated a significant laboratory portion [2]. In these labs, students become familiar with and perform experiments on two educational rigs from Feedback, Inc. The first rig, the 38-600 Thermal Process Rig, is a thermal system in which the temperature of the inlet and outlet flow can be controlled by varying the flow rate through a heat exchanger. The second rig, the 38-100 Basic Process Rig, allows students to control flow and level throughout a fluid system. This project serves to improve upon the current set up of this lab by more closely associating the experiments with material presented in lecture and in other classes, including Systems Engineering and Fluid Mechanics.

II. SYSTEM OVERVIEW

The system under consideration for this project is the Feedback Basic Process Rig 38-100 [3]. The set up, shown in Fig. 1, includes the right side of the tank, a solenoid valve, a servo valve, a centrifugal pump, a visual flow meter, and copper piping. The right side of the tank has a cross-sectional area of 205 cm². An empty tank corresponds to 0% level as designated by the ruler on the left side. The tank at 100% level is 22.25 cm above the outlet valve. The solenoid valve, with an inner diameter of 6 mm, is open for all experiments. An 8.9 cm copper pipe runs between the bottom of the tank and the solenoid valve. The position of the gate within the servo valve is controlled by sending a varying current signal between 4 mA and 20 mA, where 4 mA corresponds to the valve being 0% open, and 20 mA corresponds to the valve

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being 100% open. The servo valve controls the inlet flow rate, which is driven through the cooper pipes by the centrifugal pump. The visual flow meter, which measures the inlet flow rate, is able to read flow rates between about 1.0 L/min and 4.0 L/min in 0.2 L/min increments.

III. Calculating System Time Constant

A. Method 1: Initial tank level free response

From Systems Dynamics theory, by definition, the time constant, $\tau$, is the length of time the system takes to reach 63% of steady-state. The first method used to evaluate $\tau$ involved filling the tank to 100% level and, with the pump turned off, allowing the tank to drain. Using a stopwatch, the time for the system to reach 63% of steady-state, or 37% level, was recorded to be 57 seconds.

B. Method 2: Determining resistance from head versus flow rate relationship

For a fluid system in the form:

$$A \frac{dh}{dt} + \frac{1}{R} h = q_{in}$$

(1)

where $A$ is the cross-sectional area of the tank, $R$ is the resistance of the outlet valve, and $h$ is the head, or level of fluid in the tank, the time constant, $\tau$, can be calculated by the following:

$$\tau = RA$$

(2)

The area, $A$, can be measured, and the resistance, $R$, can be determined experimentally. The graph in Fig. 2 (b) shows the general relationship between head and flow rate for a fluid system, such as the one depicted in Fig. 2 (a), with a single tank, single inlet flow and single outlet flow through a valve. The resistance, $R$, of the system is equal to the slope [4]. As the head and flow rate increase, so does the resistance of the outlet valve. The slope of the graph is smaller in the laminar flow range and increases greatly as the system becomes turbulent.

Fig. 3 shows the experimental results for the system under consideration by allowing the tank level to come to steady-state for a span of flow rates between 1.25 L/m and 1.7 L/m. Lower flow rates were not considered due to the deadband of the visual flow meter at these flow rates. Therefore, only the higher flow rates are represented in Fig. 3. The data was graphed using Microsoft Excel, and a linear trendline was placed through the data points ($y_b$) yielding a slope which represents the average resistance for the experimental operational points. Another trendline was placed through the data points with a y-intercept at zero ($y_a$), which biases the slope, and corresponding resistance, but takes into account that the flow rate is zero when the head is zero.
L/min to 1.7 L/min, or roughly \(20 \frac{\text{cm}^3}{\text{s}}\) to \(27 \frac{\text{cm}^3}{\text{s}}\), yields a time constant of 133 seconds.

C. Method 3: Linearizing About a Single Operational Point

From [5], Reynolds number is defined as,

\[
Re = \frac{\nu D_{\text{pipe}}}{v}
\]

where \(\nu\) is kinematic viscosity, \(\nu\) is the flow velocity and \(D_{\text{pipe}}\) is the diameter of the pipe. Assuming a kinematic viscosity of \(1.0 \times 10^6 \frac{\text{m}^2}{\text{s}}\) for water, the Reynolds number ranges from 2000 to 2800 for the system. This is in the transitional zone between laminar and turbulent, and the system is modeled as turbulent.

From [4], the output flow rate of a turbulent system can be described by,

\[
q_{out} = k \sqrt{h}
\]

where \(k\) is a flow coefficient, measured in \(\text{m}^2/\text{s}\), and \(h\) is, again, level.

A simplified model of fluid system is:

\[
A \frac{dh}{dt} = q_{in} - q_{out}
\]

Combining these two equations yield:

\[
A \frac{dh}{dt} + k \sqrt{h} = q_{in}
\]

To linearize around a nominal operating point, use Taylor Series neglecting higher-order terms,

\[
f(h) = f(\bar{h}) + \frac{df}{dh} |_{h=\bar{h}} (h - \bar{h})
\]

where the head is equal to the nominal head, \(\bar{h}\), and the incremental head, \(\dot{h}\),

\[
h = \bar{h} + \dot{h}
\]

The nominal inlet flow rate, \(\bar{q}_{in}\), and incremental inlet flow rate, \(\dot{q}_{in}\), are defined by:

\[
q_{in} = \bar{q}_{in} + \dot{q}_{in}
\]

This yields:

\[
f(h) = k \sqrt{\bar{h}} + \frac{k}{2 \sqrt{\bar{h}}} \dot{h}
\]

Inserting (9)-(10) into (6) yields,

\[
A \frac{d(\bar{h} + \dot{h})}{dt} + k \sqrt{\bar{h}} + \frac{k}{2 \sqrt{\bar{h}}} \dot{h} = \bar{q}_{in} + \dot{q}_{in}
\]

Inserting the equilibrium condition and recognizing \(\frac{d\bar{h}}{dt} = 0\) yields the governing equation around the nominal values of \(\bar{h}\) and \(\bar{q}_{in}\),

\[
A \frac{d\bar{h}}{dt} + \frac{k}{2 \sqrt{\bar{h}}} \bar{h} = q_{in}
\]

Noting that

\[
k = \frac{\bar{q}_{in}}{\sqrt{\bar{h}}}
\]

yields

\[
A \frac{d\bar{h}}{dt} + \frac{\bar{q}_{in}}{2 \sqrt{\bar{h}}} \bar{h} = \bar{q}_{in}
\]

By converting this equation to the form:

\[
\tau \frac{dh}{dt} + \bar{h} = 0
\]

it is apparent that:

\[
\tau = \frac{2A\bar{h}}{\bar{q}_{in}}
\]

for a given operation point in the turbulent range.

Considering the nominal operational point, \(\bar{q}_{in}\), of 1.45 L/m, or \(23.7 \frac{\text{cm}^3}{\text{s}}\), results in a steady-state level of 48%, or 15.31 cm head, and yields a time constant of 265 seconds.

\[
\tau = \frac{2(205 \text{ cm}^2)(15.31 \text{ cm})}{23.7 \frac{\text{cm}^3}{\text{s}}} = 265 \text{ seconds}
\]

D. Method 4: Energy Considerations in Pipe Flow

The engineering Bernoulli’s equation,

\[
\left(\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + h_1\right) - \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_2\right) = H_T
\]

can be used to determine energy losses within a fluid system, where subscripts 1 and 2 refer to the values of density, fluid velocity and head at the designated points 1 and 2, and \(H_T\) refers to total head loss, due to both major and minor losses.

For the system under consideration, point 1 is the top level of 133 seconds.

\[
\text{point 1 is}(100\%)\text{ in the tank and point 2 directly under the outlet valve, as shown in Fig. 4.}
\]

The major losses in the system, due to friction losses in the pipes, occur in the pipe between the tank and the outlet valve, and can be determined by the following equation:

\[
H_{\text{major}} = f \frac{L}{D_{\text{pipe}}} \frac{V_2^2}{2g}
\]
where $f$ is the friction factor, and $L$ is the length of the piping between the tank and the valve. The friction factor, $f$, is a function of Reynolds number and roughness, $e$, divided by the pipe diameter, $D_{\text{pipe}}$. For copper piping, $e$ is 0.0015. The average Reynolds number for the system, from (3), is 2300. Using a Moody chart, the friction factor is found to be about 0.046 [5].

From [4], for turbulent flow,

$$2H = Rq_{in} \quad (24)$$

By rearranging (23) into this format,

$$2H = \left( K_v + 1 + f \frac{L}{D_{\text{pipe}}} \right) \left( \frac{16 q_{in}}{\pi^2 D_{\text{pipe}}^4 g} \right) q_{in} \quad (25)$$

it is apparent that

$$R = \left( K_v + 1 + f \frac{L}{D_{\text{pipe}}} \right) \left( \frac{16 q_{in}}{\pi^2 D_{\text{pipe}}^4 g} \right) \quad (26)$$

$$R = (81.86 + 1 + (0.046) \frac{8.9 \text{ cm}}{1.26 \text{ cm}^3}) \left( \frac{16 (23.7 \text{ cm}^3)}{\pi^2 (1.26 \text{ mm})^4 (9.81 \text{ m/s}^2)} \right)$$

$$R = 13180 \frac{s}{m^2} = 1.32 \frac{s}{cm^2}$$

From (2), the system time constant determined by this method is 270 seconds.

$$\tau = (1.32 \text{ seconds/cm}^2) (205 \text{ cm}^3) = 270 \text{ seconds}$$

E. Method 5: Initial condition with inlet flow rate yielding steady-state level of 50%

Methods 2, 3 and 4 look at operating points around higher flow rates, where the resistance is roughly linear. Method 1 looks at the operating range of the entire system, which is overall highly nonlinear. Due to the roughly linear behavior in this system across higher flow rates, as show in Fig. 2(b), a time constant determined experimentally for operational points across higher flow rates only is expected to more closely match a time constant determined analytically by assuming linearity in this range, such as in Methods 2 and 3. An experiment similar to Method 1 was run, looking only at the operating range between 100% to 50% level. Similar to Method 1, the tank was filled to 100% level as an initial condition. Instead of observing the zero input response, an input flow rate of 1.5 L/m (23.8 cm³/s), for which the steady-state value is 50% level, was supplied to the tank. To measure the time constant, the time for the tank to drain from 100% level to 66% level, or 63% to steady-state, was timed. By this method, $\tau$ was determined to be 230 seconds.

IV. RESULTS AND DISCUSSION

The time constants determined by each method are listed in Table I. It is interesting that both linearization, a technique taught in Systems Engineering, and an energy loss consideration, a method taught in Fluid Mechanics, yielded similar results around a single operational point. The time constant found experimentally in Method 5 varies from these time constants, but is still similar. This validates both linearization and the energy loss consideration, as both methods are reasonable approximations of the system.
Methods 1 and 2 yielded far different results for the system time constants, most prominently because these methods are looking at a larger range of operational points. As Fig. 2(b) shows, $R$ changes drastically depending on the flow rate and level the system is being operated around. Method 1 considers all values of $R$, as it requires the system to go from 100% level to 0% level. Theoretically, the value of $\tau$ from Method 2A, using the slope of the trendline which goes through the origin, should yield a similar value as Method 1. However, due to the measurement deadband of the visual flow meter at low input flow rates, there is data missing from the system representation in Fig. 3. If this data could be collected, it would likely change the linear trendline calculated by Microsoft Excel, which would in turn change the resistance value and time constant calculated by this method. Still, there would be a discrepancy between the values, as Method 2A assumes a linear system, which the system is obviously not.

<table>
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<tr>
<th>Method</th>
<th>Time constant, $\tau$ (seconds)</th>
</tr>
</thead>
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<tr>
<td>1: Initial tank level free response</td>
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</tr>
<tr>
<td>2A: Determining resistance from head versus flow rate relationship</td>
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</tr>
<tr>
<td>(y-intercept=0)</td>
<td></td>
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<td>2B: Determining resistance from head versus flow rate relationship</td>
<td>334</td>
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<td>3: Linearizing around a single operational point</td>
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<tr>
<td>4: Utilizing energy considerations in pipe flow</td>
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<td>230</td>
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</tbody>
</table>

The value of $\tau$ calculated using the slope of the trendline through the experimental data points collected in Method 2B is higher than the time constants determined by Methods 3, 4, and 5. Method 2B considers the resistance value for all flow rates recorded, from 1.25 L/m to 1.7 L/m, or roughly $20 \frac{cm^3}{s}$ to $27 \frac{cm^3}{s}$. As seen in Fig. 2, the graph has an increasing slope in this range, and therefore the average value of $\tau$ would be larger than the value of $\tau$ at 50% level. Theoretically, if Method 5 was repeated going from 100% level to a steady-state of 80% level instead of 50% level, the value of $\tau$ recorded would be closer to the value calculated in Method 2.

**V. INTEGRATION INTO COURSE MATERIAL**

All of the methods used in this project to determine the time constant involve concepts learned in Systems Engineering, Fluid Mechanics or Feedback Control Systems. System Engineering asks students to consider linearization of a system around an operational point to model system responses. Fluid Mechanics gives students tools to consider energy losses in pipe flows. Feedback Control Systems offers students hands-on experience with physical systems so that they may compare modeled behavior with actual behavior.

The current proposal for this project asks students to complete Method 3 as a homework assignment in Systems Engineering class, given the experimental data for the operational point at a flow rate of 1.45 L/m and the diagram of the system shown in Fig. 4. The following semester, in Fluid Mechanics, the students will derive an additional model of the system, following Method 4, again looking at the operation point at a flow rate of 1.45 L/m. Concurrently, in Feedback Control Systems, students will experimentally determine $\tau$, using Methods 1 and 2 using the 38-100 Basic Process Rig. They will be asked to think about why there are discrepancies between the two values and hopefully realize the consequences of modeling a nonlinear system as linear. After comparing the results, students will use the experimental values of $\tau$ to choose PID control gains that achieve the desired liquid-level response. In this way, students validate the accuracy of their models with experimentally determined values of the time constant, $\tau$, and implement their findings in a real-time control system. Synthesizing concepts from different mechanical engineering courses and allowing students the opportunity to experiment with and validate these concepts aims to provide students with a more concrete understanding of the theory they learn.

**REFERENCES**


